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Transformations of Non-terminating Basic Hypergeometric Series, Their Contour Integrals and Applications to Rogers–Ramanujan Identities

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1. INTRODUCTION

The transformation theory of the generalised basic hypergeometric series using q-analogue of Barnes' contour integral of the first kind was developed by Watson [23]. Later, Sears [11–15], in a series of remarkable papers, developed a very general transformtion theory of both ordinary and basic hypergeometric series. Sears' work was simplified to a considerable extent by Slater [18], who used a special kind of contour integrals. Agarwal and Verma [2] defined the bi-basic hypergeometric series and used the contour integral of the Slater type to obtain some general transformations between bi-basic hypergeometric series.

The work of Watson [24] and its subsequent extensions by Bailey [7, 8] and Slater [16, 17], using the transformation theory of basic hypergeometric series, have proved to be the key of the analytic side of the partition theory. It was pointed out by Andrews [3] that in all the identities of Rogers–Ramanujan type due to Bailey [7, 8] and Slater [16, 17] involve only the prime factors 2, 3, 5, 7. He posed and solved the problem of finding identities related to modulus 11. The key result of Andrews from which he deduced identities of modulus 11 is

$$
{}_8\phi_6\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, iq^{-n}, -iq^{-n}, -q^{-n}, q^{-n}, 0; q; -aq^{1+4n} \\ \sqrt{a}, -\sqrt{a}, -aiq^{1+n}, iaq^{1+n}, -aq^{1+n}, aq^{1+n} \end{matrix}\right]
= \frac{[aq;q]_{4n}}{[a^4q^{4+4n};q^4]_n}\, {}_3\phi_2\left[\begin{matrix} -q^{2n}, q^{-2n}, 0; q^2; q^2 \\ \dfrac{q^{-4n}}{a}, \dfrac{q^{1-4n}}{a} \end{matrix}\right], \tag{1.1}
$$

which he obtained using q-difference equations. Recently the authors [21]

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have developed some transformations between terminating bi-basic hypergeometric series, which not only yielded (1.1) as a special case but also provided a key for deducing transformations, which when specialised yielded identities on modulus 13, 17, 19 etc.

It may be pertinent to note that the recent rapid advance of Lie algebra starting with Macdonald's [10] major breakthrough has produced significant results illustrating the relationship between Lie algebras and partition identities. A proof of the weak form of a beautiful conjecture of Macdonald concerning rationally invariant plane partitions by Andrews [4] relies on the transformation between terminating nearly-poised ${}_4F_3$ into a Saalschützian ${}_5F_4$ due to Whipple. A q-analogue of Whipple transformation was developed by the authors [22] to enable one to attack the complete Macdonald conjecture.

In view of the applications of the transformations of terminating basic hypergeometric series developed by the authors [21] to the analytical side of the identities of Rogers–Ramanujan type, it is natural to expect that a transformation theory between the non-terminating bi-basic hypergeometric series would be a great asset in deducing analytical identities of Rogers–Ramanujan type.

In this note, using the method of summation of a series of lower order, non-terminating versions of the bi-basic transformations proved earlier in [21] are developed. Alternative proofs of these results are also discussed by using the q-analogue of Barnes' contour integral of the first kind due to Watson [23] and that of second kind due to Agarwal [1]. Using contour integrals we also obtain the non-terminating version of the transformations of Bailey [6] and Jain [9] connecting nearly-poised and well-poised basic hypergeometric series.

The paper is concluded by deducing some analytical identities of Rogers–Ramanujan type on modulus 39, 51, 57, 66, 78, 102 and 114 as applications of the results obtained in Sections 3 and 4.

2. Definitions and Notations

If we let $|q| < 1$, $[a;q]_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, $[a;q]_0 = 1$, $[a;q]_\infty = \prod_{j=0}^{\infty}(1-aq^j)$ then we may define the basic hypergeometric series as

$$
{}_{p+1}\phi_{p+r}\left[\begin{matrix} a_1, a_2, \ldots, a_{p+1}; q; x \\ b_1, b_2, \ldots, b_{p+r} \end{matrix}\right]
= \sum_{n=0}^{\infty} \frac{[a_1;q]_n [a_2;q]_n \cdots [a_{p+1};q]_n (-)^{nr} x^n q^{r(n/2)(n-1)}}{[q;q]_n [b_1;q]_n \cdots [b_{p+r};q]_n},
$$

where the series ${}_{p+1}\phi_{p+r}$ converges for all positive integral values of r and for all x, except when $r=0$ it converges only for $|x|<1$.

The series

$$ {}_{p+1}\phi_{p+r}\left[\begin{matrix} q^{a_1},\dots,q^{a_{p+1}};q;x \\ q^{b_1},\dots,q^{b_{p+r}} \end{matrix}\right] $$

shall be denoted by ${}_{p+1}\mathscr{F}_{p+r}\left[\begin{matrix} a_1,a_2,\dots,a_{p+1};q;x \\ b_1,b_2,\dots,b_{p+r} \end{matrix}\right]$ whenever there is a parameter of the type $-q^{a_m}$ in ${}_{p+1}\phi_{p+r}$ it will be denoted by $(a_m)^*$ in ${}_{p+1}\mathscr{F}_{p+r}$. Further, we shall denote by $\Pi\left[\begin{matrix} a_1,a_2,\dots,a_r;q \\ b_1,b_2,\dots,b_s \end{matrix}\right]$ the infinite product

$$ \prod_{j=0}^{\infty}\left[\frac{(1-a_1q^j)(1-a_2q^j)\cdots(1-a_rq^j)}{(1-b_1q^j)(1-b_2q^j)\cdots(1-b_sq^j)}\right] $$

and shall abbreviate the infinite product

$$ \prod_{i=0}^{\infty}\left[\frac{(1-q^{a_1+j})(1-q^{a_2+j})\cdots(1-q^{a_r+i})}{(1-q^{b_1+i})(1-q^{b_2+j})\cdots(1-q^{b_s+i})}\right] $$

by $\mathscr{P}\left[\begin{matrix} a_1,a_2,\dots,a_r;q \\ b_1,b_2,\dots,b_s \end{matrix}\right]$. Whenever there is a factor of the type $\prod_{j=0}^{\infty}(1+q^{a_m+j})$ in the product, we shall denote it by $(a_m)^*$ in the $\mathscr{P}$-symbol.

The well-poised series

$$ {}_{p+3}\phi_{p+2}\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b_1,b_2,\dots,b_p;q;z \\ \sqrt{a},-\sqrt{a},\dfrac{aq}{b_1},\dfrac{aq}{b_2},\dots,\dfrac{aq}{b_p} \end{matrix}\right] $$

would be abbreviated by ${}_{p+3}W_{p+2}|a;b_1,b_2,\dots,b_p;q;z|$, whereas the well-poised basic hypergeometric series

$$ {}_{p+3}\mathscr{F}_{p+2}\left[\begin{matrix} a,1+\dfrac{a}{2},\left(1+\dfrac{a}{2}\right)^*,b_1,\dots,b_p;q;z \\ \dfrac{a}{2},\left(\dfrac{a}{2}\right)^*,1+a-b_1,\dots,1+a-b_p \end{matrix}\right] $$

would be denoted by ${}_{p+3}\Omega_{p+2}|a;b_1,b_2,\dots,b_p;q;z|$.

3.

We begin this section by proving the transformation

$$
{}_{10}W_9\left[a; b, x, -x, y, -y, z, -z; q; \frac{-a^3q^3}{bx^2y^2z^2}\right]
= \Pi\begin{bmatrix} a^2q^2, \frac{a^2q^2}{x^2y^2}, \frac{a^2q^2}{x^2z^2}, \frac{a^2q^2}{y^2z^2}; q^2 \\ \frac{a^2q^2}{x^2}, \frac{a^2q^2}{y^2}, \frac{a^2q^2}{z^2}, \frac{a^2q^2}{x^2y^2z^2} \end{bmatrix}
\cdot {}_5\phi_4\begin{bmatrix} x^2, y^2, z^2, \frac{-aq}{b}, \frac{-aq^2}{b}; q^2; q^2 \\ \frac{x^2y^2z^2}{a^2}, \frac{a^2q^2}{b^2}, -aq, -aq^2 \end{bmatrix}
$$

$$
+ \Pi\begin{bmatrix} x^2, y^2, z^2, \frac{a^4q^4}{b^2x^2y^2z^2}; q^2 \\ \frac{a^2q^2}{x^2}, \frac{a^2q^2}{y^2}, \frac{a^2q^2}{z^2}, \frac{x^2y^2z^2}{a^2q^2} \end{bmatrix}
\Pi\begin{bmatrix} aq, \frac{-a^3q^3}{x^2y^2z^2}; q \\ \frac{aq}{b}, \frac{-a^3q^3}{bx^2y^2z^2} \end{bmatrix}
\cdot {}_5\phi_4\begin{bmatrix} \frac{a^2q^2}{x^2y^2}, \frac{a^2q^2}{x^2z^2}, \frac{a^2q^2}{y^2z^2}, \frac{-a^3q^3}{bx^2y^2z^2}, \frac{-a^3q^4}{bx^2y^2z^2}; q^2; q^2 \\ \frac{a^2q^4}{x^2y^2z^2}, \frac{a^4q^4}{b^2x^2y^2z^2}, \frac{-a^3q^3}{x^2y^2z^2}, \frac{-a^3q^4}{x^2y^2z^2} \end{bmatrix}, \tag{3.1}
$$

which is a non-terminating version of a known result [21; 1.3].

Proof of (3.1) *by Summing Series of Lower Order.* In the non-terminating version of Watson's q-analogue of Whipple transformation [19; 3.4.2.5], setting $e = 1$, we get

$$
\Pi\begin{bmatrix} aq, \frac{aq}{fg}, \frac{aq}{fh}, \frac{aq}{gh}; q \\ f, g, h, \frac{aq}{fgh} \end{bmatrix}
{}_3\phi_2\begin{bmatrix} f, g, h; q; q \\ aq, \frac{fgh}{a} \end{bmatrix}
+ \Pi\begin{bmatrix} \frac{a^2q^2}{fgh}; q \\ \frac{fgh}{aq} \end{bmatrix}
{}_3\phi_2\begin{bmatrix} \frac{aq}{gh}, \frac{aq}{fh}, \frac{aq}{fg}; q; q \\ \frac{aq^2}{fgh}, \frac{a^2q^2}{fgh} \end{bmatrix}
= \Pi\begin{bmatrix} \frac{aq}{f}, \frac{aq}{g}, \frac{aq}{h}; q \\ f, g, h \end{bmatrix}. \tag{3.2}
$$

In (3.2) first replacing q by q^2 and then a, f, g, h by $a^2q^{4r}, x^2q^{2r}, y^2q^{2r}, z^2q^{2r}$, respectively, multiplying both sides by

$$\frac{[a;q]_r\,[aq^2;q^2]_r\,[b;q]_r(-)^r(aq)^{3r}}{[q;q]_r\,[a;q^2]_r\left[\dfrac{aq}{b};q\right]_r(bx^2y^2z^2)^r}$$

and summing with respect to r from 0 to ∞, we get

$$\Pi\begin{bmatrix} a^2q^2, \dfrac{a^2q^2}{x^2y^2}, \dfrac{a^2q^2}{x^2z^2}, \dfrac{a^2q^2}{y^2z^2}; q^2 \\ x^2, y^2, z^2, \dfrac{a^2q^2}{x^2y^2z^2} \end{bmatrix}$$

$$\cdot \sum_{n=0}^{\infty} \frac{[x^2;q^2]_n\,[y^2;q^2]_n\,[z^2;q^2]_n q^{2n}}{[q^2;q^2]_n\,[a^2q^2;q^2]_n\left[\dfrac{x^2y^2z^2}{a^2};q^2\right]_n}$$

$$\cdot\,{}_6W_5\left[a;b,-q^{-n},q^{-n};q;\frac{-aq^{1+2n}}{b}\right]+\Pi\begin{bmatrix}\dfrac{a^4q^4}{x^2y^2z^2};q^2\\ x^2y^2z^2/a^2q^2\end{bmatrix}$$

$$\cdot\sum_{n=0}^{\infty}\frac{\left[\dfrac{a^2q^2}{x^2y^2};q^2\right]_n\left[\dfrac{a^2q^2}{x^2z^2};q^2\right]_n}{[q^2;q^2]_n\left[\dfrac{a^2q^4}{x^2y^2z^2};q^2\right]_n}\cdot\frac{\left[\dfrac{a^2q^2}{y^2z^2};q^2\right]_n q^{2n}}{\left[\dfrac{a^4q^4}{x^2y^2z^2};q^2\right]_n}$$

$$\cdot\,{}_6W_5\left[a;b,\frac{xyz}{a}q^{-n-1},\frac{-xyz}{a}q^{-n-1};q;\frac{-a^3q^{3+2n}}{bx^2y^2z^2}\right]$$

$$=\Pi\begin{bmatrix}\dfrac{a^2q^2}{x^2},\dfrac{a^2q^2}{y^2},\dfrac{a^2q^2}{z^2};q^2\\ x^2,y^2,z^2\end{bmatrix}$$

$$\cdot\,{}_{10}W_9\left[a;b,x,-x,y,-y,z,-z;q;\frac{-a^3q^3}{bx^2y^2z^2}\right].$$

Now summing the ${}_6W_5$'s on the left-hand side of the above expression, we get (3.1) on some simplification.

Proof of (3.1) *by Using q-Analogues of Barnes Contour Integrals.* Using the q-analogue of Barnes' second lemma due to Agarwal [1; 3.5], viz.,

$$\frac{1}{2\pi i}\int_C \mathscr{P}\begin{bmatrix} 1+s,\, 1+a+s,\, \beta+s;\, q_1 \\ x+s,\, y+s,\, z+s \end{bmatrix} \frac{\pi q_1^s\, ds}{\sin \pi s \sin \pi(1-\beta-s)}$$

$$= \operatorname{cosec} \pi(\beta-1)\, \mathscr{P}\begin{bmatrix} 1, \beta, 1-\beta, 1+a-x, 1+a-y, 1+a-z;\, q_1 \\ x, y, z, 1+x-\beta, 1+y-\beta, 1+z-\beta \end{bmatrix}, \tag{3.3}$$

provided that $|q_1| < 1$, $\beta = x+y+z-a$, the integral being convergent for $\operatorname{Re}[s \log q_1 - \log \sin \pi s \sin \pi(\beta+s)] < 0$, when $|s|$ is large, it is easy to verify that

$$\operatorname{cosec} \pi(\beta-1)\, \mathscr{P}\begin{bmatrix} 1, \beta+n, 1-\beta-n, 1+a-x+n, 1+a-y+n, 1+a-z+n;\, q_1 \\ x+n, y+n, z+n, 1+a-x-y, 1+a-y-z, 1+a-x-z \end{bmatrix}$$

$$= \frac{1}{2\pi i}\int_C \mathscr{P}\begin{bmatrix} 1+a+s, x+y+z-a+s, 1+s;\, q_1 \\ x+s, y+s, z+s \end{bmatrix}$$

$$\cdot \frac{[q_1^{-s}; q_1]_n (-)^n \pi q_1^{s(1+n)-(1/2)n(n+1)}}{[q_1^{1+a+s}; q_1]_n \sin \pi s \sin \pi(1-\beta-s)}\, ds, \tag{3.4}$$

where $\beta = x+y+z-a$. In (3.4) replacing n by jn (where j is a positive integer), multiplying both sides by

$$\frac{[q_2^{C_1}; q_2]_n \cdots [q_2^{C_r}; q_2]_n}{[q_2^{d_1}; q_2]_n \cdots [q_2^{d_r}; q_2]_n}\, q^{nt+(1/2)nj(nj-1)}(-)^{nj},$$

summing from $n=0$ to ∞, and on the right-hand side interchanging the order of summation and integration, we get

$$\operatorname{cosec} \pi(\beta-1)\, \mathscr{P}\begin{bmatrix} 1, \beta, 1-\beta, 1+a-x, 1+a-y, 1+a-z;\, q_1 \\ x, y, z, 1+a-x-y, 1+a-y-z, 1+a-x-z \end{bmatrix}$$

$$\cdot \sum_{n=0}^{\infty} \frac{[q_1^x; q_1]_{nj} [q_1^y; q_1]_{nj} [q_1^z; q_1]_{nj} [q_2^{C_1}; q_2]_n \cdots [q_2^{C_r}; q_2]_n q_1^{n(t-\beta j)}}{[q_1^{1+a-x}; q_1]_{nj} [q_1^{1+a-y}; q_1]_{nj} [q_1^{1+a-z}; q_1]_{nj} [q_2^{d_1}; q_2]_n \cdots [q_2^{d_r}; q_2]_n}$$

$$= \frac{1}{2\pi i}\int_C \mathscr{P}\begin{bmatrix} 1+a+s, x+y+z-a+s, 1+s;\, q_1 \\ x+s, y+s, z+s \end{bmatrix} \frac{\pi q_1^s}{\sin \pi s \sin \pi(1-\beta-s)}$$

$$\cdot \left\{ \sum_{n=0}^{\infty} \frac{[q_2^{C_1}; q_2]_n \cdots [q_2^{C_r}; q_2]_n [q_1^{-s}; q_1]_{nj}}{[q_2^{d_1}; q_2]_n \cdots [q_2^{d_r}; q_2]_n [q_1^{1+a+s}; q_1]_{nj}}\, q_1^{n(t-j+js)} \right\} ds, \tag{3.5}$$

where $\beta = x+y+z-a$, $|q_2| < 1$.

However, in (3.5) first setting $j=1$, $q_1 = q^2$, $q_2 = q$, $r=4$ and then $C_1 = a$, $C_2 = 1+a/2$, $C_3 = b$, $q^{C_4} = -q^{1+a/2}$, $d_1 = 1$, $d_2 = a/2$, $d_3 = 1+a-b$, $q^{d_4} = -q^{a/2}$, $q^{2t} = -q^{3+a-b}$ and summing the inner well-poised ${}_6\phi_5$ on the right-hand side by [10; 3.3.13], we get

$$\operatorname{cosec} \pi(\beta - 1) \cdot \mathscr{P}\left[\begin{matrix} 1, \beta, 1-\beta, 1+a-x, 1+a-y, 1+a-z; q^2 \\ x, y, z, 1+a-x-y, 1+a-y-z, 1+a-x-z \end{matrix}\right]$$

$$\cdot\ {}_{10}\Omega_9[a; b, x, (x)^*, y, (y)^*, z, (z)^*; q; -q^{3+3a-b-2(x+y+z)}]$$

$$= \frac{1}{2\pi i} \int_C \mathscr{P}\left[\begin{matrix} 1+a, x+y+z-a+s, 1+s, 1+a-b+s; q^2 \\ x+s, y+s, z+s, 1+a-b \end{matrix}\right]$$

$$\times \mathscr{P}\left[\begin{matrix} (1+a+2s)^*, (1+a-b)^*; q \\ (1+a)^*, (1+a-b+2s)^* \end{matrix}\right] \frac{\pi q^{2s}}{\sin \pi s \sin \pi(1-\beta-s)}\, ds, \quad (3.6)$$

where $\beta = x+y+z-a$. Evaluating the integral on the right-hand side of (3.6) by considering the residues at the poles of $\operatorname{cosec} \pi s$ and $\operatorname{cosec} \pi(1-\beta-s)$ lying to the right of the contour C, we get (3.1) on replacing q^a, q^b, q^x, q^y, q^z by a, b, c, x, y and z, respectively (see Watson [23] for details).

Special Cases

(i) In (3.1) replacing "a" by "$-a$," we get the q-analogue of a result of Bailey [5: 6.5(1)].

(ii) In (3.1) replacing "a" by "$-a$" and setting $z = q^{-n}$, we get the q-analogue of a formula of Whipple [25: 2.5] in the form

$${}_{10}W_9\left[-a; b, x, -x, y, -y, -q^{-n}, q^{-n}; q; \frac{a^3 q^{3+2n}}{bx^2y^2}\right]$$

$$= \frac{[a^2q^2; q^2]_n \left[\frac{a^2q^2}{x^2y^2}; q^2\right]_n}{\left[\frac{a^2q^2}{x^2}; q^2\right]_n \left[\frac{a^2q^2}{y^2}; q^2\right]_n}\ {}_5\phi_4\left[\begin{matrix} x^2, y^2, \frac{aq}{b}, \frac{aq^2}{b}, q^{-2n}; q^2; q^2 \\ aq, aq^2, a^2q^2/b^2, \frac{x^2y^2q^{-2n}}{a^2} \end{matrix}\right]. \quad (3.7)$$

However, (3.7) for $n \to \infty$, yields the q-analogue of another formula of Whipple [25; 2.5] in the form

$$\sum_{m=0}^{\infty} \frac{[-a; q]_m [-aq^2; q^2]_m [b; q]_m [x^2; q^2]_m [y^2; q^2]_m}{[q; q]_m [-a; q^2]_m \left[\frac{-aq}{b}; q\right]_m \left[\frac{a^2q^2}{x^2}; q^2\right]_m \left[\frac{a^2q^2}{y^2}; q^2\right]_m}$$

$$\cdot\ q^{m(m+2)} \left(\frac{-a^3}{bx^2y^2}\right)^m$$

$$= \Pi \left[\begin{matrix} a^2q^2, \frac{a^2q^2}{x^2y^2}; q^2 \\ \frac{a^2q^2}{x^2}, \frac{a^2q^2}{y^2} \end{matrix}\right]\ {}_4\phi_3\left[\begin{matrix} x^2, y^2, \frac{aq}{b}, \frac{aq^2}{b}; q^2; \frac{a^2q^2}{x^2y^2} \\ aq, aq^2, a^2q^2/b^2 \end{matrix}\right]. \quad (3.8)$$

In this sequel it is worthwhile to remark that further special cases of (3.8) may be discussed on the lines of Whipple [25].

(iii) In (3.7) setting $x^2=aq$, $y^2=aq^2$ and then summing the resulting ${}_3\phi_2$ on the right-hand side by the q-analogue of Saalschütz summation theorem [19; 3.3.2.2], we get

$$\sum_{m=0}^{n} \frac{[-a;q]_m [a^2q^4;q^4]_m [b;q]_m [q^{-2n};q^2]_m q^{2mn}}{[q;q]_m [a^2;q^4]_m \left[\frac{-aq}{b};q\right]_m [a^2q^{2+2n};q^2]_m} \left(\frac{a}{b}\right)^m$$
$$= \frac{[a^2q^2;q^2]_n \left[\frac{a}{b};q\right]_{2n}}{\left[\frac{a^2q^2}{b^2};q^2\right]_n [a;q]_{2n}}, \tag{3.9}$$

which is the q-analogue of the terminating version of a result of Bailey [5; 6.4(2)].

(iv) Lastly, using (3.9) we prove the transformation:

$${}_{12}W_{11}\left[a; iq\sqrt{a}, -iq\sqrt{a}, b, x, -x, y, -y, z, -z; q; \frac{a^3q^2}{bx^2y^2z^2}\right]$$
$$= \Pi\begin{bmatrix} \frac{a^2q^2}{x^2y^2}, \frac{a^2q^2}{x^2z^2}, \frac{a^2q^2}{y^2z^2}, a^2q^2; q^2 \\ \frac{a^2q^2}{x^2}, \frac{a^2q^2}{y^2}, \frac{a^2q^2}{z^2}, \frac{a^2q^2}{x^2y^2z^2} \end{bmatrix} {}_5\phi_4\begin{bmatrix} x^2, y^2, z^2, \frac{-a}{b}, \frac{-aq}{b}; q^2; q^2 \\ \frac{x^2y^2z^2}{a^2}, \frac{a^2q^2}{b^2}, -a, -aq \end{bmatrix}$$
$$+ \Pi\begin{bmatrix} x^2, y^2, z^2, a^2q^2, \frac{a^4q^4}{b^2x^2y^2z^2}; q^2 \\ \frac{a^2q^2}{x^2}, \frac{a^2q^2}{y^2}, \frac{a^2q^2}{z^2}, \frac{a^2q^2}{b^2}, \frac{x^2y^2z^2}{a^2q^2} \end{bmatrix} \Pi\begin{bmatrix} \frac{-a}{b}, \frac{-a^3q^2}{x^2y^2z^2}; q \\ -a, \frac{-a^3q^2}{bx^2y^2z^2} \end{bmatrix}$$
$$\cdot {}_5\phi_4\begin{bmatrix} \frac{a^2q^2}{x^2y^2}, \frac{a^2q^2}{y^2z^2}, \frac{a^2q^2}{x^2z^2}, \frac{-a^3q^2}{bx^2y^2z^2}, \frac{-a^3q^3}{bx^2y^2z^2}; q; q \\ \frac{a^4q^4}{b^2x^2y^2z^2}, \frac{a^2q^4}{x^2y^2z^2}, \frac{-a^3q^2}{x^2y^2z^2}, \frac{-a^3q^3}{x^2y^2z^2} \end{bmatrix}. \tag{3.10}$$

It may be remarked that (3.10) with "a" replaced by "$-a$" yields a q-analogue of a formula of Bailey [5; 6.4(3)] whereas on replacing "a" by "$-a$" and setting $z=q^{-n}$, gives us a q-analogue of a transformation of Bailey [5; 4.6(2)].

Proof of (3.10). In (3.5) first setting $j=1$, $q_1=q^2$, $q_2=q$, $r=6$ and then

$C_1 = a$, $C_2 = 1 + a/2$, $q^{C_3} = -q^{1+a/2}$, $q^{C_4} = iq^{1+a/2}$, $q^{C_5} = -iq^{1+a/2}$, $C_6 = b$, $d_1 = 1$, $d_2 = a/2$, $q^{d_3} = -q^{a/2}$, $q^{d_4} = iq^{a/2}$, $q^{d_5} = -iq^{a/2}$, $d_6 = 1 + a - b$, $q^{2t} = -q^{2+a-b}$, then summing the resulting series on the right-hand side by (3.9), we get

$$\operatorname{cosec} \pi(\beta - 1) \cdot \mathscr{P}\left[\begin{matrix} 1, \beta, 1-\beta, 1+a-x, 1+a-y, 1+a-z; q^2 \\ x, y, z, 1+a-x-y, 1+a-y-z, 1+a-x-z \end{matrix}\right]$$
$$\times {}_{12}W_{11}[q^a; iq^{1+a/2}, -iq^{1+a/2}, q^b, q^x, -q^x,$$
$$q^y, -q^y, q^z, -q^z; q; -q^{2+3a-b-2(x+y+z)}]$$
$$= \frac{1}{2\pi i}\int_C \mathscr{P}\left[\begin{matrix} 1+a, x+y+z-a+s, 1+s, 1+a-b+s; q^2 \\ x+s, y+s, z+s, 1+a-b \end{matrix}\right]$$
$$\times \mathscr{P}\left[\begin{matrix} (a-b)^*, (a+2s)^*; q \\ (a)^*, (a-b+2s)^* \end{matrix}\right] \cdot \frac{\pi q^{2s}}{\sin \pi s \sin \pi(1-\beta-s)}\, ds,$$

where $\beta = x + y + z - a$. This gives (3.10) on evaluating the integral on the right-hand side of the above expression by considering the residues at the poles of $\operatorname{cosec} \pi s$ and $\operatorname{cosec} \pi(1 - \beta - s)$ lying to the right of the contour C and finally replacing q^a, q^b, q^x, q^y, q^z by a, b, x, y, z, respectively.

4.

In this section we prove the transformation

$$ {}_{10}W_9\left[a; b, x, xq, y, yq, z, zq; q^2; \frac{a^3q^3}{bx^2y^2z^2}\right]$$
$$= \Pi\left[\begin{matrix} aq, \frac{aq}{xy}, \frac{aq}{xz}, \frac{aq}{yz}; q \\ \frac{aq}{x}, \frac{aq}{y}, \frac{aq}{z}, \frac{aq}{xyz} \end{matrix}\right] {}_5\phi_4\left[\begin{matrix} x, y, z, \sqrt{\frac{aq}{b}}, -\sqrt{\frac{aq}{b}}; q; q \\ \frac{xyz}{a}, \frac{aq}{b}, \sqrt{aq}, -\sqrt{aq} \end{matrix}\right]$$
$$+ \Pi\left[\begin{matrix} \frac{a^3q^3}{x^2y^2z^2}, aq^2; q^2 \\ \frac{aq^2}{b}, \frac{a^3q^3}{bx^2y^2z^2} \end{matrix}\right] \Pi\left[\begin{matrix} x, y, z, \frac{a^2q^2}{bx^2yz}; q \\ \frac{aq}{x}, \frac{aq}{y}, \frac{aq}{z}, \frac{xyz}{aq} \end{matrix}\right]$$
$$\times {}_5\phi_4\left[\begin{matrix} \frac{aq}{xy}, \frac{aq}{yz}, \frac{aq}{xz}, \sqrt{\frac{a^3q^3}{bx^2y^2z^2}}, -\sqrt{\frac{a^3q^3}{bx^2y^2z^2}}; q; q \\ \frac{aq^2}{xyz}, \frac{a^2q^2}{bxyz}, \sqrt{\frac{a^3q^3}{x^2y^2z^2}}, -\sqrt{\frac{a^3q^3}{x^2y^2z^2}} \end{matrix}\right], \tag{4.1}$$

which for $z = q^{-n}$ reduces to a known transformation [21; 1.4].

Proof of (4.1) *by Summing Series of Lower Order.* In (3.2) replacing a, f, g, h by aq^{4r}, xq^{2r}, yq^{2r}, zq^{2r}, multiply both sides by

$$\frac{[a;q^2]_r\,[aq^4;q^4]_r\,[b;q^2]_r}{[q^2;q^2]_r\,[a;q^4]_r\left[\dfrac{aq^2}{b};q^2\right]_r}\left(\frac{a^3q^3}{bx^2y^2z^2}\right)^r$$

and summing with respect to r from 0 to ∞, we get

$$\Pi\begin{bmatrix} aq,\dfrac{aq}{xy},\dfrac{aq}{xz},\dfrac{aq}{yz};q \\ x,y,z,\dfrac{aq}{xyz} \end{bmatrix}\sum_{n=0}^{\infty}\frac{[x;q]_n\,[y;q]_n\,[z;q]_n\,q^n}{[q;q]_n\,[aq;q]_n\left[\dfrac{xyz}{a};q\right]_n}$$

$$\cdot\,{}_6W_5\left[a;b,q^{-n},q^{-n+1};q^2;\frac{aq^{1+2n}}{b}\right]$$

$$+\,\Pi\begin{bmatrix} \dfrac{a^2q^2}{xyz};q \\ xyz/aq \end{bmatrix}\sum_{n=0}^{\infty}\frac{\left[\dfrac{aq}{xy};q\right]_n\left[\dfrac{aq}{xz};q\right]_n\left[\dfrac{aq}{yz};q\right]_n}{[q;q]_n\left[\dfrac{a^2q^2}{xyz};q\right]_n\left[\dfrac{aq^2}{xyz};q\right]_n}q^n$$

$$\cdot\,{}_6W_5\left[a;b,\frac{xyz}{a}q^{-n},\frac{xyz}{a}q^{-n-1};q^2;\frac{a^3q^{3+2n}}{bx^2y^2z^2}\right]$$

$$=\Pi\begin{bmatrix} \dfrac{aq}{x},\dfrac{aq}{y},\dfrac{aq}{z};q \\ x,y,z \end{bmatrix}\,{}_{10}W_9\left[a;b,x,xq,y,yq,z,zq;q^2;\frac{a^3q^3}{bx^2y^2z^2}\right].$$

Now summing the two well-poised ${}_6\phi_5$'s on the left-hand side of the above expression, we get (4.1) on some simplification.

Proof of (4.1) *by q-Analogue of Barnes Contour Integral.* In (3.5) first setting $j=2$, $q_1=q$, $q_2=q^2$, $r=4$ and then $C_1=a/2$, $C_2=1+a/4$, $C_3=b/2$, $q^{2C_4}=-q^{2+a/2}$, $d_1=1$, $d_2=a/4$, $d_3=1+a/2-b/2$, $q^{2d_4}=-q^{a/2}$, $t=3+a-b$, summing the inner series on the right-hand side by [19; 3.3.1.4], we get

$$\operatorname{cosec}\pi(\beta-1)\,\mathscr{P}\begin{bmatrix} 1,\beta,1-\beta,1+a-x,1+a-y,1+a-z;q \\ x,y,z,1+a-x-y,1+a-y-z,1+a-x-z \end{bmatrix}$$

$$\cdot\,{}_{10}\Omega_9\left[\frac{a}{2};\frac{b}{2},\frac{x}{2},\frac{1+x}{2},\frac{y}{2},\frac{1+y}{2},\frac{z}{2},\frac{1+z}{2};q^2;q^{3+3a-b-2(x+y+z)}\right]$$

$$= \frac{1}{2\pi i}\int_C \mathscr{P}\left[\begin{matrix} x+y+z-a+s, 1+s, 1+a-b+s, 1+a, (1+a)^*, \\ \frac{1+a}{2}+s, \left(\frac{1+a}{2}+s\right)^*; q \\ x+s, y+s, z+s, \frac{2+a-b}{2}, \left(\frac{2+a-b}{2}\right)^*, \\ \frac{1+a-b}{2}+s, \left(\frac{1+a-b}{2}+s\right)^* \end{matrix}\right]$$

$$\cdot \frac{\pi q^s}{\sin \pi s \sin \pi(1-\beta-s)}\,ds,$$

where $\beta = x+y+z-a$. Evaluating the integral on the right-hand side of the above expression by considering the residues at the poles of $\operatorname{cosec} \pi s$ and $\operatorname{cosec} \pi(1-\beta-s)$ lying to the right of the contour C, we get (4.1) (on replacing q^a, q^b, q^x, q^y, q^z by a, b, x, y, z, respectively).

5.

We begin this section by proving a non-terminating version of a known formula [21; 1.5] in the form

$$\begin{aligned} &{}_{12}W_{11}\left[a; x, \omega x, \omega^2 x, y, \omega y, \omega^2 y, z, \omega z, \omega^2 z; q; \frac{a^4q^4}{x^3y^3z^3}\right] \\ &= \Pi\left[\begin{matrix} a^3q^3, \frac{a^3q^3}{x^3y^3}, \frac{a^3q^3}{x^3z^3}, \frac{a^3q^3}{y^3z^3}; q^3 \\ \frac{a^3q^3}{x^3}, \frac{a^3q^3}{y^3}, \frac{a^3q^3}{z^3}, \frac{a^3q^3}{x^3y^3z^3} \end{matrix}\right] \\ &\quad \cdot {}_6\phi_5\left[\begin{matrix} x^3, y^3, z^3, aq, aq^2, aq^3; q^3; q^3 \\ \frac{x^3y^3z^3}{a^3}, (aq)^{3/2}, -(aq)^{3/2}, q^3a^{3/2}, -q^3a^{3/2} \end{matrix}\right] \\ &\quad + \Pi\left[\begin{matrix} x^3, y^3, z^3, \frac{a^9q^9}{(xyz)^6}; q^3 \\ \frac{a^3q^3}{x^3}, \frac{a^3q^3}{y^3}, \frac{a^3q^3}{z^3}, \frac{(xyz)^3}{a^3q^3} \end{matrix}\right] \frac{[aq;q]_\infty}{\left[\frac{a^4q^4}{(xyz)^3};q\right]_\infty} \\ &\quad \cdot {}_6\phi_5\left[\begin{matrix} \frac{a^3q^3}{x^3y^3}, \frac{a^3q^3}{x^3z^3}, \frac{a^3q^3}{y^3z^3}, \frac{a^4q^4}{(xyz)^3}, \frac{a^4q^5}{(xyz)^3}, \frac{a^4q^6}{(xyz)^3}; q^3; q^3 \\ \frac{a^3q^6}{(xyz)^3}, \frac{(aq)^{9/2}}{(xyz)^3}, \frac{-(aq)^{9/2}}{(xyz)^3}, \frac{a^{9/2}q^6}{(xyz)^3}, \frac{-a^{9/2}q^6}{(xyz)^3} \end{matrix}\right], \end{aligned} \tag{5.1}$$

where ω is the cube root of unity.

Proof of (5.1) *by Summing Series of Lower Order.* In (3.2) first replacing q by q^3 and then a, f, g, h by $a^3q^{6r}, x^3q^{3r}, y^3q^{3r}, z^3q^{3r}$, respectively, multiplying both the sides by

$$\frac{[a;q]_r\,[aq^2;q^2]_r}{[q;q]_r\,[a;q^2]_r}\left(\frac{a^4q^4}{x^3y^3z^3}\right)^r$$

and summing with respect to r from 0 to ∞, we get

$$\Pi\begin{bmatrix} a^3q^3, \dfrac{a^3q^3}{x^3y^3}, \dfrac{a^3q^3}{x^3z^3}, \dfrac{a^3q^3}{y^3z^3}; q^3 \\ x^3, y^3, z^3, \dfrac{a^3q^3}{(xyz)^3} \end{bmatrix}$$

$$\sum_{n=0}^{\infty} \frac{[x^3;q^3]_n\,[y^3;q^3]_n\,[z^3;q^3]_n\,q^{3n}}{[q^3;q^3]_n\,[a^3q^3;q^3]_n\left[\left(\dfrac{xyz}{a}\right)^3;q^3\right]_n}$$

$${}_6W_5[a;q^{-n},\omega q^{-n},\omega^2q^{-n};q;aq^{1+3n}]$$

$$+\,\Pi\begin{bmatrix} \dfrac{a^6q^6}{(xyz)^3}; q^3 \\ \left(\dfrac{xyz}{aq}\right)^3 \end{bmatrix}$$

$$\times\sum_{n=0}^{\infty} \frac{\left[\left(\dfrac{aq}{xy}\right)^3;q^3\right]_n\left[\left(\dfrac{aq}{xz}\right)^3;q^3\right]_n\left[\left(\dfrac{aq}{yz}\right)^3;a^3\right]_n}{[q^3;q^3]_n\left[\left(\dfrac{a^2q^2}{xyz}\right)^3;q^3\right]_n\left[\left(\dfrac{aq^2}{xyz}\right)^3;q^3\right]_n}\,q^{3n}$$

$${}_6W_5\left[a;\frac{xyzq^{-n-1}}{a},\frac{\omega xyzq^{-n-1}}{a},\frac{\omega^2xyzq^{-n-1}}{a};q;\frac{a^4q^{4+3n}}{(xyz)^3}\right]$$

$$=\Pi\begin{bmatrix} (aq/x)^3, \left(\dfrac{aq}{y}\right)^3, \left(\dfrac{aq}{z}\right)^3; q^3 \\ x^3, y^3, z^3 \end{bmatrix}$$

$${}_{12}W_{11}\left[a;x,\omega x,\omega^2x,\,y,\omega y,\omega^2y,z,\omega z,\omega^2z;q;\frac{a^4q^4}{(xyz)^3}\right].$$

Now, summing the two well-poised ${}_6\phi_5$'s on the left-hand side of the above expression, we get (5.1).

Proof of (5.1) *Using q-Analogues of Barnes Contour Integral.* In (3.5) first setting $j=1$, $q_1=q^3$, $q_2=q$, $r=3$ and then $C_1=a$, $C_2=1+a/2$,

$d_1 = 1$, $d_2 = a/2$, $q^{C_3} = -q^{1+a/2}$, $q^{d_3} = -q^{a/2}$, $t = \frac{1}{3}(a+4)$, summing the inner series on the right-hand side by [19; 3.3.14], we get

$$\operatorname{cosec} \pi(\beta - 1) \cdot \mathscr{P}\left[\begin{matrix} 1, \beta, 1-\beta, 1+a-x, 1+a-y, 1+a-z; q^3 \\ x, y, z, 1+a-x-y-z, 1+a-y-z, 1+a-x-z \end{matrix}\right]$$

$$\cdot {}_{12}W_{11}[q^a; q^x, \omega q^x, \omega^2 q^x, q^y, \omega q^y, \omega^2 q^y, q^z, \omega q^z, \omega^2 q^z; q; q^{4+4a-3(x+y+z)}]$$

$$= \frac{1}{2\pi i}\int_C \mathscr{P}\left[\begin{matrix} x+y+z-a+s, 1+s, 1+a+2s; q^3 \\ x+s, y+s, z+s \end{matrix}\right]$$

$$\cdot \frac{[q^{1+a}; q]_\infty}{[q^{1+a+3s}; q]_\infty} \frac{\pi q^{3s}\, ds}{\sin \pi s \sin \pi(1-\beta-s)},$$

where $\beta = x + y + z - a$. Evaluating the integral on the right-hand side of the above expression by considering the residues at the poles of cosec πs and cosec $\pi(1-\beta-s)$ lying to the right of the contour C, we get (5.1) on replacing q^a, q^x, q^y, q^z by a, x, y, z, respectively).

6.

In this section we prove the non-terminating version of a known result [21; 1.6] in the form:

$$\begin{aligned} &{}_{12}W_{11}\left[a; x, xq, xq^2, y, yq, yq^2, z, zq, zq^2; q^3; \frac{a^4q^3}{(xyz)^3}\right] \\ &= \Pi\left[\begin{matrix} aq, \dfrac{aq}{xy}, \dfrac{aq}{xz}, \dfrac{aq}{yz}; q \\ \dfrac{aq}{x}, \dfrac{aq}{y}, \dfrac{aq}{z}, \dfrac{aq}{xyz} \end{matrix}\right] \\ &\quad\cdot {}_6\phi_5\left[\begin{matrix} x, y, z, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}; q; q \\ \dfrac{xyz}{a}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix}\right] \\ &+ \Pi\left[\begin{matrix} x, y, z, \dfrac{a^3q^2}{(xyz)^2}; q \\ \dfrac{dq}{x}, \dfrac{aq}{y}, \dfrac{aq}{z}, \dfrac{xyz}{aq} \end{matrix}\right] \Pi\left[\begin{matrix} aq^3; q^3 \\ \dfrac{a^4q^3}{(xyz)^3} \end{matrix}\right] \\ &\quad\cdot {}_6\phi_5\left[\begin{matrix} \dfrac{aq}{xy}, \dfrac{aq}{xz}, \dfrac{aq}{yz}, \dfrac{a^{4/3}q}{xyz}, \dfrac{a^{4/3}\omega q}{xyz}, \dfrac{a^{4/3}\omega^2 q}{xyz}; q; q \\ \dfrac{aq^2}{xyz}, \dfrac{a^{3/2}q}{xyz}, \dfrac{-a^{3/2}q}{xyz}, \dfrac{(aq)^{3/2}}{xyz}, \dfrac{-(aq)^{3/2}}{xyz} \end{matrix}\right]. \end{aligned} \tag{6.1}$$

Proof of (6.1) *by Summing Series of Lower Order.* In (3.2) replacing a, f, g, h by aq^{6r}, xq^{3r}, yq^{3r}, zq^{3r}, respectively, multiplying both sides by

$$\frac{[a;q^3]_r\,[aq^6;q^6]_r\,a^{4r}q^{3r}}{[a^3;q^3]_r\,[a,q^6]_r\,(xyz)^{3r}}$$

and summing with respect to r from 0 to ∞, we get

$$\begin{aligned}
&\Pi\begin{bmatrix} aq, \dfrac{aq}{xy}, \dfrac{aq}{xz}, \dfrac{aq}{yz}; q \\ x, y, z, aq/xyz \end{bmatrix} \\
&\quad\cdot \sum_{n=0}^{\infty} \frac{[x;q]_n\,[y;q]_n\,[z;q]_n}{[q;q]_n \left[\dfrac{xyz}{a};q\right]_n [aq;q]_n}\, q^n \\
&\quad\cdot {}_6W_5[a;q^{-n},q^{1-n},q^{2-n};q^3;aq^{3n}] \\
&\quad+\Pi\begin{bmatrix} \dfrac{a^2q^2}{xyz}; q \\ \dfrac{xyz}{aq} \end{bmatrix} \sum_{n=0}^{\infty} \frac{\left[\dfrac{aq}{xy};q\right]_n \left[\dfrac{aq}{xz};q\right]_n \left[\dfrac{aq}{yz};q\right]_n}{[q;q]_n \left[\dfrac{a^2q^2}{xyz};q\right]_n \left[\dfrac{aq^2}{xyz};q\right]_n}\, q^n \\
&\quad\cdot {}_6W_5\left[a;\frac{xyz}{a}q^{-n-1},\frac{xyz}{a}q^{-n},\frac{xyz}{a}q^{1-n};q^3;\frac{a^4q^{3+3n}}{(xyz)^3}\right] \\
&=\Pi\begin{bmatrix} \dfrac{aq}{x}, \dfrac{aq}{y}, \dfrac{aq}{z}; q \\ x, y, z \end{bmatrix} \\
&\quad\cdot {}_{12}W_{11}\left[a;x,xq,xq^2,y,yq,yq^2,z,zq,zq^2;q^3;\frac{a^4q^3}{x^3y^3z^3}\right].
\end{aligned}$$

Summing the two well-poised ${}_6\phi_5$'s on the left-hand side of the above expression, we get (6.1).

Proof of (6.1) *Using Basic Barnes Contour Integrals.* In (3.5) first setting $j=3$, $q_1=q$, $q_2=q^3$, $r=3$ and then $3C_1=a$, $6C_2=6+a$, $d_1=1$, $6d_2=a$, $q^{3C_3}=-q^{3+a/2}$, $q^{3d_3}=-q^{a/2}$, $t=a+3$, summing the inner series on the right-hand side by [19; 3.3.1.4], we get

$$\operatorname{cosec}\pi(\beta-1)\cdot\mathscr{P}\left[\begin{matrix}1,\beta,1-\beta,1+a-x,1+a-y,1+a-z;q\\ x,y,z,1+a-x-y,1+a-y-z,1+a-x-z\end{matrix}\right]$$

$$\cdot{}_{12}\Omega_{11}\left[\frac{a}{3};\frac{x}{3},\frac{1+x}{3},\frac{2+x}{3},\frac{y}{3},\frac{1+y}{3},\frac{2+y}{3},\frac{z}{3},\frac{1+z}{3},\frac{2+z}{3};\quad q^3;q^{3+4a-3(x+y+z)}\right]$$

$$=\frac{1}{2\pi i}\int_C\mathscr{P}\left[\begin{matrix}x+y+z-a+s,1+s,a+2s;q\\ x+s,y+s,z+s\end{matrix}\right]$$

$$\cdot\frac{[q^{3+a};q^3]_\infty}{[q^{a+3s};q^3]_\infty}\cdot\frac{\pi q^s}{\sin\pi s\sin\pi(1-\beta-s)}\,ds,$$

where $\beta=x+y+z-a$. Evaluating the integrals on the right of the above expression by considering the residues at the poles of $\operatorname{cosec}\pi s$ and $\operatorname{cosec}\pi(1-\beta-s)$ lying to the right of the contour C, yield (6.1) (on replacing q^a, q^x, q^y, q^z by a, x, y, z, respectively).

7.

We begin this section by proving a non-terminating version of a formula of Bailey [6] (incidently the result obtained is a q-analogue of a result of Bailey [5; 6.9(1)]), $k=a^2q/cde$

$$\Pi\left[\begin{matrix}\frac{aq}{c},\frac{aq}{d},\frac{aq}{e},\frac{a^2f}{k^2};q\\ a,c,d,e,f\end{matrix}\right]{}_5\phi_4\left[\begin{matrix}a,c,d,e,f;q;q\\ \frac{aq}{c},\frac{aq}{d},\frac{aq}{e},\frac{a^2f}{k^2}\end{matrix}\right]$$

$$-q^{1+2k-2a-f}\Pi\left[\begin{matrix}\frac{k^2q^2}{afc},\frac{k^2q^2}{adf},\frac{k^2q^2}{aef},\frac{k^2q^2}{a^2f};q\\ \frac{k^2q}{af},\frac{k^2qc}{a^2f},\frac{k^2dq}{a^2f},\frac{k^2eq}{a^2f},\frac{k^2q}{a^2}\end{matrix}\right]$$

$$\cdot{}_5\phi_4\left[\begin{matrix}\frac{k^2q}{a^2},\frac{k^2qc}{a^2f},\frac{k^2qd}{a^2f},\frac{k^2qe}{a^2f},\frac{k^2q}{af};q;q\\ \frac{k^2q^2}{afc},\frac{k^2q^2}{adf},\frac{k^2q^2}{aef},\frac{k^2q^2}{a^2f}\end{matrix}\right]$$

$$= \Pi \begin{bmatrix} \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{kq}{f}, \frac{kq}{a}, \frac{k^2q}{a}, \frac{a^2f}{k^2}, \frac{k^2q}{a^2f}; q \\ c, d, e, \frac{kq}{af}, \frac{k^2q}{a^2}, kq, a, f, \frac{k^2q}{af} \end{bmatrix}$$

$$\cdot {}_{12}W_{11}\left[k; \frac{kc}{a}, \frac{kd}{a}, \frac{ke}{a}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, f, \frac{k^2q}{af}; q; q\right]$$

$$- \Pi \begin{bmatrix} \frac{kq^2}{cf}, \frac{kq^2}{df}, \frac{kq^2}{ef}, \frac{kc}{a}, \frac{kd}{a}, \frac{ke}{a}, \frac{k^2q}{a^2f}, \frac{k^4q^3}{a^3f^2}, \frac{a^2f}{k^2}; q \\ c, d, e, \frac{af}{kq}, \frac{k^2q}{a^2}, \frac{k^2q}{af}, \frac{k^2cq}{a^2f}, \frac{k^2dq}{a^2f}, \frac{k^2eq}{a^2f}, \frac{k^3q^3}{a^2f^2} \end{bmatrix}$$

$$\cdot {}_{12}W_{11}\left[\frac{k^3q^2}{a^2f^2}; \frac{k^2cq}{a^2f}, \frac{k^2dq}{a^2f}, \frac{k^2eq}{a^2f}, \frac{kq}{f\sqrt{a}}, -\frac{kq}{f\sqrt{a}}, \frac{kq^{3/2}}{f\sqrt{a}}, -\frac{kq^{3/2}}{f\sqrt{a}}, \frac{kq}{a}, \frac{k^2q}{af}; q; q\right]. \tag{7.1}$$

Proof of (7.1). In the integral analogue of the basic Dougall theorem due to Agarwall [1; 5.6]

$$\operatorname{cosec} \pi(a-b) \mathscr{P} \begin{bmatrix} 1, 1+a-b, b-a, 1+a-d-e, 1+a-e-f, 1+a-f-d, \\ 1+a-c-d, 1+a-c-e, 1+a-c-f; q \\ b, c, d, e, f, b+c-a, b+d-a, b+e-a, b+f-a \end{bmatrix}$$

$$= \frac{1}{2\pi i} \int_C \mathscr{P} \begin{bmatrix} 1+s, a/2+s, (a/2+s)^*, 1+a-b+s, 1+a-c+s, \\ 1+a-d+s, 1+a-e+s, 1+a-f+s; q \\ a+s, 1+a/2+s, (1+a/2+s)^*, b+s, \\ c+s, d+s, e+s, f+s \end{bmatrix}$$

$$\cdot \frac{\pi q^s}{\sin \pi s \sin \pi(b-a-s)} ds. \tag{7.2}$$

(Provided that $1+2a = b+c+d+e+f$), writing $k = 1+2a-c-d-e$ and then replacing a, c, d, e, k, b by k, $k+c-a$, $k+d-a$, $k+e-a$, a, $a+t$, multiplying by

$$\frac{1}{2\pi i} \mathscr{P} \begin{bmatrix} b-k-t, 1-b+k+t, \alpha_1+t, \alpha_2+t; q \\ b-a-t, f+t, \beta_1+t, \beta_2+t \end{bmatrix} \frac{1}{\sin \pi(1-b+k+t)}$$

integrating with respect to t from $-i\pi/z$ to $i\pi/z$ (where $q = e^{-z}$, $z > 0$) and then writing $t+s$ for t on the right-hand side, we have

$$\frac{1}{2\pi i}\int_{-i\pi/z}^{i\pi/z} \mathscr{P}\begin{bmatrix} 1, c, d, e, 1+k-a-t, a+t-k, 1+a-c+t, 1+a-d+t, \\ 1+a-e+t, \alpha_1+t, \alpha_2+t, b-k-t, 1-b+k+t; q \\ k+c-a, k+d-a, k+e-a, a-k, a+t, c+t, d+t \\ e+t, -t, \beta_1+t, \beta_2+t, f+t, b-a-t \end{bmatrix}$$

$$\cdot \frac{dt}{\sin \pi(k-a-t) \sin \pi(1-b+k+t)}$$

$$= \frac{1}{2\pi i}\int_C \mathscr{P}\begin{bmatrix} 1+s, k/2+s, (k/2+s)^*, 1+a-c+s, \\ 1+a-d+s, 1+a-e+s; q \\ k+s, 1+k/2+s, (1+k/2+s)^*, \\ k+c-a+s, k+d-a+s, k+e-a+s \end{bmatrix}$$

$$\cdot \frac{1}{2\pi i}\int_{-i\pi/z}^{i\pi/z} \mathscr{P}\begin{bmatrix} 1+k+2s+t, 1+k-a-t, b-k-s-t, \\ 1-b+k+s+t, \alpha_1+s+t, \alpha_2+s+t; q \\ a+t+2s, -t, b-a-s-t, \\ f+s+t, \beta_1+s+t, \beta_2+s+t \end{bmatrix}$$

$$\cdot \frac{\pi q^s \, dt \, ds}{\sin \pi s \sin \pi(a-k+t) \sin \pi(1-b+k+s+t)}.$$

Now, evaluating the integral on the left-hand side and the inner integral on the right-hand side, we obtain on some simplification

$$\mathscr{P}\begin{bmatrix} 1+a-b, 1+a-c, 1+a-d, 1+a-e, \alpha_1, \alpha_2; q \\ a, c, d, e, f, \beta_1, \beta_2 \end{bmatrix}$$

$$\cdot {}_7\mathscr{F}_6\begin{bmatrix} a, c, d, e, f, \beta_1, \beta_2; q; q \\ 1+a-c, 1+a-d, 1+a-e, 1+a-b, \alpha_1, \alpha_2 \end{bmatrix}$$

$$- q^{b-a}\mathscr{P}\begin{bmatrix} 1+b-c, 1+b-d, 1+b-e, 1+b-a, \alpha_1+b-a, \\ \alpha_2+b-a; q \\ b, b+c-a, b+d-a, b+e-a, b+f-a, \\ \beta_1+b-a, \beta_2+b-a \end{bmatrix}$$

$$\cdot {}_7\mathscr{F}_6\begin{bmatrix} b, b+c-a, b+d-a, b+e-a, b+f-a, \\ \beta_1+b-a, \beta_2+b-a; q; q \\ 1+b-c, 1+b-d, 1+b-e, 1+b-a, \\ \alpha_1+b-a, \alpha_2+b-a \end{bmatrix}$$

$$= \sin \pi(k-b) \mathscr{P}\begin{bmatrix} k+c-a, k+d-a, k+e-a, 1+a-b, b-a, a-k; q \\ 1, c, d, e, b-k, 1-b+k \end{bmatrix}$$

$$\cdot \frac{1}{2\pi i}\int_C \mathscr{P}\begin{bmatrix} 1+s, k/2+s, (k/2+s)^*, 1+a-c+s, 1+a-d+s, \\ 1+a-e+s, b-k-s, 1-b+k+s; q \\ k+s, 1+k/2+s, (1+k/2+s)^*, k+c-a+s, \\ k+d-a+s, k+e-a+s, b-a-s, 1+a-b+s \end{bmatrix}$$

$$\cdot \Big\{ \mathscr{P}\begin{bmatrix} 1+k+2s, 1+a-b+s, \alpha_1+s, \alpha_2+s; q \\ a+2s, f+s, a-k, \beta_1+s, \beta_2+s \end{bmatrix}$$

$$\cdot {}_5\mathscr{F}_4\begin{bmatrix} a+2s, a-k, f+s, \beta_1+s, \beta_2+s; q; q \\ 1+k+2s, 1+a-b+s, \alpha_1+s, \alpha_2+s \end{bmatrix}$$

$$- q^{b-a-s}\, \mathscr{P}\begin{bmatrix} 1+k+b-a+s, 1+b-a-s, \alpha_1+b-a, \alpha_2+b-a; q \\ b-k-s, b+s, f+b-a, \beta_1+b-a, \beta_2+b-a \end{bmatrix}$$

$$\cdot {}_5\mathscr{F}_4\begin{bmatrix} b+s, f+b-a, b-k-s, \beta_1+b-a, \beta_2+b-a; q; q \\ 1+k+b-a+s, 1+b-a-s, \alpha_1+b-a, \alpha_2+b-a \end{bmatrix} \Big\}$$

$$\cdot \frac{\pi q^s}{\sin \pi s \sin \pi(b-k-s)}\, ds. \tag{7.3}$$

In (7.3) setting $\alpha_1 = \alpha_2 = \beta_1 = \beta_2$, $b = 1 + 2k - a - f$, summing the resulting ${}_3\mathscr{F}_2$'s on the right-hand side by a formula of Sears [12; 5.2], and raplacing a, c, d, e, f, k by q^a, q^c, q^d, q^e, q^f, q^k (S denoting the left-hand side of (7.1)), we get

$$S = \sin \pi(k-a-f)$$

$$\mathscr{P}\begin{bmatrix} k+c-a, k+d-a, k+e-a, 1+2k-2a-f, 2a-2k+f, 1+k-a; q \\ c, d, e, 1+k-a-f, a+f-k, 1, 1+2k-2a \end{bmatrix}$$

$$\cdot \frac{1}{2\pi i}\int_C$$

$$\mathscr{P}\begin{bmatrix} 1+s, k/s+s, (k/2+s)^*, 1+a-c+s, 1+a-d+s, \\ 1+a-e+s, a+f-k+s, 1+k-f+s, 1+2k-a+2s; q \\ k+s, 1+k/2+s, (1+k/s+s)^*, k+c-a+s, \\ k+d-a+s, k+e-a+s, a+2s, f+s, 1+2k-a-f+s \end{bmatrix}$$

$$\cdot \frac{\pi q^s\, ds}{\sin \pi s \sin \pi(1+k-a-f-s)}. \tag{7.4}$$

Evaluating the integral on the right of (7.4) by considering the residues at the poles of cosec πs and cosec $\pi(1+k-a-f-s)$ lying to the right of the contour C, we get (7.1) (on replacing q^a, q^c, q^d, q^e, q^f, q^k by a, c, d, e, f, k, respectively).

However, in (7.4) setting $d = 1/2 + a/2$, $e = 1 + a/2$, $k = a - c - 1/2$ then evaluating the resulting integral on the right-hand side by (7.2), we get

$$\Pi\begin{bmatrix} \frac{aq}{c}, \sqrt{a}, fc^2q; q \\ a, c, q\sqrt{a}, f \end{bmatrix} {}_4\phi_3\begin{bmatrix} a, q\sqrt{a}, c, f; q; q \\ \sqrt{a}, \frac{aq}{c}, c^2fq \end{bmatrix}$$

$$-\frac{1}{c^2f}\Pi\begin{bmatrix} \frac{aq}{fc^3}, \frac{\sqrt{a}}{c^2f}, \frac{q}{c^2f}; q \\ \frac{a}{c^2f}, \frac{1}{cf}, \frac{q\sqrt{a}}{c^2f}, \frac{1}{c^2} \end{bmatrix} {}_4\phi_3\begin{bmatrix} \frac{a}{c^2f}, \frac{1}{cf}, \frac{q\sqrt{a}}{c^2f}, \frac{1}{c^2}; q; q \\ \frac{\sqrt{a}}{c^2f}, \frac{aq}{fc^3}, \frac{q}{c^2f} \end{bmatrix}$$

$$= \Pi\begin{bmatrix} \frac{\sqrt{a}}{c}, \frac{\sqrt{aq}}{c}, \frac{1}{c^2f}, c^2fq, \frac{1}{c}, \left(\frac{\sqrt{aq}}{c}\right)^*, \left(\frac{q\sqrt{a}}{c}\right)^*, \left(\frac{\sqrt{a}}{cf}\right)^*, \frac{aq}{cf}; q \\ c, \sqrt{aq}, q\sqrt{a}, \frac{1}{c^2}, \frac{a}{c^2f}, f, (\sqrt{a})^*, (\sqrt{aq})^*, \left(\frac{q\sqrt{a}}{cf}\right)^*, \frac{1}{cf} \end{bmatrix}. \tag{7.5}$$

Formula (7.5) is the non-terminating version of a formula of Jain [9; 1.5]. On the one hand, replacing $\sqrt{a}$ by $-\sqrt{a}$ in (7.5) we get the q-analogue of the non-terminating version of a known result [5; 4.5(1.2)].

On the other hand in (7.4) first replacing c by $(x)^*$ and setting $d = 1 + a/2$, $e = (1 + a/2)^*$, $k = a - c - 1$, and then evaluating the resulting integral on the right-hand side by (7.2), we get (on replacing x by $(c)^*$)

$$\Pi\begin{bmatrix} \frac{aq}{c}, \sqrt{a}, -\sqrt{a}, c^2fq^2; q \\ a, c, q\sqrt{a}, -q\sqrt{a}, f \end{bmatrix} {}_5\phi_4\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, c, f; q; q \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{c}, fc^2q^2 \end{bmatrix}$$

$$-\frac{1}{fqc^2}\Pi\begin{bmatrix} \frac{a}{fc^3}, \frac{\sqrt{a}}{fqc^2}, -\frac{\sqrt{a}}{fqc^2}, \frac{1}{c^2f}; q \\ \frac{a}{fqc^2}, \frac{1}{cfq}, \frac{\sqrt{a}}{fc^2}, -\frac{\sqrt{a}}{fc^2}, \frac{1}{c^2q} \end{bmatrix}$$

$$\cdot {}_5\phi_4\begin{bmatrix} \frac{1}{c^2q}, \frac{\sqrt{a}}{fc^2}, -\frac{\sqrt{a}}{fc^2}, \frac{1}{cfq}, \frac{a}{fqc^2}; q; q \\ \frac{\sqrt{a}}{fqc^2}, -\frac{\sqrt{a}}{fqc^2}, \frac{a}{fc^3}, \frac{1}{fc^2} \end{bmatrix}$$

$$= \Pi \begin{bmatrix} \frac{1}{c^2qf}, fc^2q^2, \frac{1}{cq}, \frac{1}{cf}\sqrt{\frac{a}{q}}, -\frac{1}{cf}\sqrt{\frac{a}{q}}, \frac{a}{c^2}, \frac{aq}{cf}; q \\ c, \frac{1}{c^2q}, \frac{a}{fqc^2}, f, \frac{\sqrt{aq}}{cf}, -\frac{\sqrt{aq}}{cf}, aq, \frac{1}{cfq} \end{bmatrix}. \tag{7.6}$$

Formula (7.6) is the non-terminating version of a result due to Bailey [6].
However, in (7.3) setting $b = 2k - a - f$, $\beta_1 = 1 + a/2$, $\alpha_1 = a/2$, $\alpha_2 = \beta_2$ and then summing the resulting ${}_4\mathscr{F}_3$ on the right-hand side by (7.5), we get

$$\mathscr{P}\begin{bmatrix} 1+a-c, 1+a-d, 1+a-e, 1+2a-2k+f, \frac{a}{2}; q \\ a, c, d, e, f, 1+\frac{a}{2} \end{bmatrix}$$

$$\cdot\, {}_6\mathscr{F}_5 \begin{bmatrix} a, 1+\frac{a}{2}, c, d, e, f; q; q \\ \frac{a}{2}, 1+a-c, 1+a-d, 1+a-e, 1+2a-2k+f \end{bmatrix}$$

$$-q^{2k-2a-f}\mathscr{P}\begin{bmatrix} 1+2k-a-f-c, 1+2k-a-f-d, 1+2k-a-f-e, \\ 1+2k-2a-f, 2k-\frac{3a}{2}-f; q \\ 2k-a-f, 2k-2a-f-c, 2k-2a-f-d, \\ 2k-2a-f-e, 2k-2a, 1+2k-\frac{3a}{2}-f \end{bmatrix}$$

$$\cdot\, {}_6\mathscr{F}_5 \begin{bmatrix} 2k-a-f, 2k-2a-f+c, 2k-2a-f-d, 2k-2a-f-e, \\ 2k-2a, 1+2k-\frac{3a}{2}-f; q; q \\ 1+2k-a-f-c, 1+2k-a-f-d, 1+2k-a-f-e, \\ 1+2k-2a-f, 2k-\frac{3a}{2}-f \end{bmatrix}$$

$$= \sin \pi(a+f-k) \mathscr{P}\begin{bmatrix} k+c-a, k+d-a, k+e-a, 1+2a-2k+f, \\ 2k-2a-f, k-a, \left(k-\dfrac{a}{2}-f\right)^{*}; q \\ 1, c, d, e, k-a-f. 1-k+a+f, \\ 2k-2a, \left(1+k-\dfrac{a}{2}-f\right)^{*} \end{bmatrix}$$

$$\cdot \frac{1}{2\pi i}\int_C$$

$$\mathscr{P}\begin{bmatrix} 1+s, \dfrac{k}{2}+s, \left(\dfrac{k}{2}+s\right)^{*}, 1+a-c+s, 1+a-d+s, 1+k-f+s, \\ 1+a-e+s, 1+a+f-k+s, \dfrac{1}{2}+k-\dfrac{a}{2}+s, k-\dfrac{a}{2}+s, \\ \left(\dfrac{1}{2}+k-\dfrac{a}{2}+s\right)^{*}, \left(1+k-\dfrac{a}{2}+s\right)^{*}; q \\ k+s, 1+\dfrac{k}{2}+s, \left(1+\dfrac{k}{2}+s\right)^{*}, k+c-a+s, f+s, \\ k+d-a+s, k+e-a+s, 2k-a-f+s, \dfrac{1}{2}+\dfrac{a}{2}+s, \\ 1+\dfrac{a}{2}+s, \left(\dfrac{1}{2}+\dfrac{a}{2}+s\right)^{*}, \left(\dfrac{a}{2}+s\right)^{*} \end{bmatrix}.$$

$$\cdot \frac{\pi q^s \, ds}{\sin \pi s \sin \pi(k-a-f-s)}. \tag{7.7}$$

Formula (7.7) is a q-analogue of a formula of Bailey [5; 6.9(3)]. This formula leads to a relation involving two Saalschützian nearly-poised series ${}_6\phi_5$ and two well-poised ${}_{12}\phi_{11}$ which are complimentary with respect to the parameter k^2/af. One of the nearly-poised series is the first kind and one is the second. When f is of the form q^{-N}, the relation reduces to a known formula [5; 4.5(4)], and when k^2e/a^2f is of the form q^{-N}, the relation reduces to the corresponding transformation of a nearly-poised series of the first kind.

It is worthwhile to remark that by replacing $\sqrt{a}$ by $-\sqrt{a}$ in (7.7), we may obtain the q-analogue of the non-terminating version of a known formula [5; 4.5(5)].

Similarly, in (7.3) setting $b = 2+2k-a-f$, $\beta_1 = 1+a/2$, $\beta_2 = (1+a/2)^*$, $\alpha_1 = a/2$, $\alpha_2 = (a/2)^*$ and then summing the resulting ${}_5\mathscr{F}_4$ on the right-hand side by (7.6), we may obtain the non-terminating version of a formula of Bailey [6].

8.

Using the results of Sections 3 and 4, identities of Rogers–Ramanujan type related to the moduli 39, 51, 57, 66, 78, 102 and 114 (in which the prime factors are 11, 13, 17 and 19) are deduced in this section. For obtaining identities of Rogers–Ramanujan type, we prove the transformations

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a^2;q^6]_{n+2r}[x;q^2]_{n+3r}[y;q^2]_{n+3r}\left[\dfrac{-aq^3}{b};q^3\right]_{2r}}{[q^6;q^6]_r[q^2;q^2]_n[-aq^3;q^3]_{2r}[a^2;q^2]_{2n+6r}\left[\dfrac{a^2q^6}{b^2};q^6\right]_r} \cdot \frac{a^{2n+8r}q^{6r(r-p+1)+2n-2pn}}{(xy)^{n+3r}}$$
$$= \Pi\begin{bmatrix} \dfrac{a^2q^2}{x}, \dfrac{a^2q^2}{y}; q^2 \\ a^2q^2, \dfrac{a^2q^2}{xy} \end{bmatrix} \sum_{j=0}^{p} \frac{[q^{-2p};q^2]_j q^{2j}}{[q^2;q^2]_j\left[\dfrac{xy}{a^2};q^2\right]_j} \cdot \sum_{n=0}^{\infty} \frac{[x;q^2]_{3n+j}}{[q^3;q^3]_n}$$
$$\cdot \frac{[y;q^2]_{3n+j}[a;q^3]_n(1-aq^{6n})[b;q^3]_n a^{9n}q^{3n(3n-2p+2)}}{(1-a)\left[\dfrac{a^2q^2}{x};q^2\right]_{3n}\left[\dfrac{a^2q^2}{y};q^2\right]_{3n}\left[\dfrac{aq^3}{b};q^3\right]_n b^n(xy)^{3n}} \tag{8.1}$$

and

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a;q^3]_{n+2r}[x;q]_{n+3r}[y;q]_{n+3r}\left[\dfrac{aq^3}{b};q^6\right]_r a^{n+4r}q^{3r(r+1-p)+n(1-p)}}{[q^3;q^3]_r[q;q]_n[aq^3;q^6]_r\left[\dfrac{aq^3}{b};q^3\right]_r[a;q]_{2n+6r}(xy)^{n+3r}}$$
$$= \Pi\begin{bmatrix} \dfrac{aq}{x}, \dfrac{aq}{y}; q \\ aq, \dfrac{aq}{xy} \end{bmatrix} \sum_{j=0}^{p} \frac{[q^{-p};q]_j q^j}{[q;q]_j\left[\dfrac{xy}{a};q\right]_j} \sum_{n=0}^{\infty} \frac{[a;q^6]_n(1-aq^{12n})}{[q^6;q^6]_n(1-a)}$$
$$\cdot \frac{[b;q^6]_n[x;q]_{6n+j}[y;q]_{6n+j}a^{9n}q^{6n(3n+1-p)}}{\left[\dfrac{aq^6}{b};q^6\right]_n\left[\dfrac{aq}{x};q\right]_{6n}\left[\dfrac{qa}{y};q\right]_{6n} b^n(xy)^{6n}}. \tag{8.2}$$

Proof of (8.1). In (3.1) replacing q by q^3 and then setting $x=q^{-n}$, $y=q^{1-n}$, $z=q^{2-n}$, we have

$$\sum_{r=0}^{[n/3]} \frac{[a;q^3]_r(1-aq^{6r})[b;q^3]_r a^{3r}q^{9r^2}}{[q^3;q^3]_r(1-a)\left[\frac{aq^3}{b};q^3\right]_r [a^2q^2;q^2]_{n+3r}[q^2;q^2]_{n-3r}b^r}$$

$$=\frac{1}{[a^2;q^2]_{2n}}\sum_{r=0}^{[n/3]} \frac{[a^2;q^6]_{n-r}\left[-\frac{aq^3}{b};q^3\right]_{2r} a^{2r}q^{6r^2}}{[q^6;q^6]_r[-aq^3;q^3]_{2r}\left[\frac{a^2q^6}{b^2};q^6\right]_r [q^2;q^2]_{n-3r}}. \tag{8.3}$$

Now, in Bailey transformation [7] setting

$$u_s=\frac{1}{[q^2;q^2]_s},\qquad v_s=\frac{1}{[a^2q^2;q^2]_s},\qquad \alpha_{3s+1}=\alpha_{3s+2}=0,$$

$$\alpha_{3s}=\frac{[a;q^3]_s(1-aq^{6s})[b;q^3]_s a^{3s}q^{9s^2}}{[q^3;q^3]_s(1-a)\left[\frac{aq^3}{b};q^3\right]_s},$$

$$\delta_s=[x;q^2]_s[y;q^2]_s\left(\frac{a^2q^{2-2p}}{xy}\right)^s$$

and evaluating $\langle\beta_n\rangle$ and $\langle\gamma_n\rangle$ by (8.3) and the formula [20]

$${}_2\phi_1\begin{bmatrix} a,b;q;\frac{ec}{ab} \\ e \end{bmatrix} = \Pi\begin{bmatrix} \frac{e}{a},\frac{e}{b};q \\ e,\frac{e}{ab} \end{bmatrix}\ {}_3\phi_2\begin{bmatrix} a,b,c;q;q \\ \frac{abq}{e},0 \end{bmatrix} \tag{8.4}$$

(where either a, b or c is of the form q^{-p}. In case only c is of the form q^{-p} then (8.4) is valid only if $|ec/ab|<1$), we get (8.1).

Proof of (8.2). In (4.1) replacing q by q^3 and then setting $x=q^{-n}$, $y=q^{1-n}$, $z=q^{2-n}$, we get

$$\sum_{r=0}^{[n/6]} \frac{[a;q^6]_r(1-aq^{12r})[b;q^6]_r a^{3r}q^{18r^2}}{[q^6;q^6]_r(1-a)\left[\frac{aq^6}{b};q^6\right]_r [aq;q]_{n+6r}[q;q]_{n-6r}b^r}$$

$$=\frac{1}{[a;q]_{2n}}\sum_{r=0}^{[n/3]} \frac{[a;q^3]_{n-r}\left[\frac{aq^3}{b};q^6\right]_r a^r q^{3r^2}}{[q^3;q^3]_r[aq^3;q^6]_r\left[\frac{aq^3}{b};q^3\right]_r [q;q]_{n-3r}}. \tag{8.5}$$

Now, in Bailey transformation [7] setting

$$u_s = \frac{1}{[q;q]_s}, \qquad v_s = \frac{1}{[aq;q]_s},$$

$$\alpha_{6s+1} = \alpha_{6s+2} = \alpha_{6s+3} = \alpha_{6s+4} = \alpha_{6s+5} = 0,$$

$$\alpha_{6s} = \frac{[a;q^6]_s(1-aq^{12s})[b;q^6]_s a^{3s} q^{18s^2}}{[q^6;q^6]_s(1-a)\left[\dfrac{aq^6}{b};q^6\right]_s b^s},$$

$$\delta_s = [x;q]_s[y;q]_s\left(\frac{aq^{1-p}}{xy}\right)^s$$

and evaluating $\langle\beta_n\rangle$ and $\langle\gamma_n\rangle$ by (8.5) and (8.4), respectively, we get (8.2).

Identities of Rogers–Ramanujan Type Related to the modulus 57

Transformation (8.1) for $b, x, y \to \infty$ reduces to

$$[a^2q^2;q^2]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a^2;q^6]_{n+2r}a^{2n+8r}q^{2n^2+12nr+24r^2-2np-6rp}}{[q^6;q^6]_r[q^2;q^2]_n[a^2;q^2]_{2n+6r}[-aq^3,q^3]_{2r}}$$
$$= \sum_{j=0}^{p}\frac{[q^{-2p};q^2]_j(-)^j a^{2j}q^{j(j+1)}}{[q^2;q^2]_j}\sum_{n=0}^{\infty}\frac{[a;q^3]_n(1-aq^{6n})(-)^n a^{9n}}{[q^3;q^3]_n(1-a)}$$
$$\cdot\, q^{(57/2)n^2-3n/2+12nj-6np}. \tag{8.6}$$

Transformation (8.6) for $a=1$ and $p=0, 1$ yields

$$[-q;q]_\infty\left\{1+\sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r-1}q^{2n^2+12nr+24r^2}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r-1}[-q^3;q^3]_{2r}}\right\}$$
$$= \prod_{n\not\equiv 0,27,30 \pmod{57}}(1-q^n)^{-1} \tag{8.7}$$

and

$$[-q;q]_\infty\left\{1+\sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r-1}q^{2n^2+12nr+24r^2-2n-6r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r-1}[-q^3,q^3]_{2r}}\right\}$$
$$= \prod_{n\not\equiv 0,21,36 \pmod{57}}(1-q^n)^{-1} + \prod_{n\not\equiv 0,24,33 \pmod{57}}(1-q^n)^{-1}. \tag{8.8}$$

Whereas, (8.6) for $a=q^3$ and $p=0, 1$ gives

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r}\, q^{2n^2+12nr+24r^2+6n+24r}}{[q^6;q^6]_r\,[q^2;q^2]_n\,[q^2;q^2]_{2n+6r+2}\,[-q^3;q^3]_{2r+1}}$$

$$= \prod_{n\not\equiv 0,3,54 \pmod{57}} (1-q^n)^{-1}. \tag{8.9}$$

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r}\, q^{2n^2+12nr+24r^2+4n+18r}}{[q^6;q^6]_r\,[q^2;q^2]_n\,[q^2;q^2]_{2n+6r+2}\,[-q^3;q^3]_{2r+1}}$$

$$= \prod_{n\not\equiv 0,9,48 \pmod{57}} (1-q^n)^{-1} - q^3 \prod_{n\not\equiv 0,3,54 \pmod{57}} (1-q^n)^{-1}. \tag{8.10}$$

Now using (8.9) in (8.10), we obtain

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r}(1+q^{2n+6r+3})\, q^{2n^2+12nr+24r^2+4n+18r}}{[q^6;q^6]_r\,[q^2;q^2]_n\,[q^2;q^2]_{2n+6r+2}\,[-q^3;q^3]_{2r+1}}$$

$$= \prod_{n\not\equiv 0,9,48 \pmod{57}} (1-q^n)^{-1}. \tag{8.11}$$

On the other hand (8.6), on setting $a=q^3$, $p=2$ and using (8.10), gives

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r}(1+q^{2n+6r+3})\, q^{2n^2+12nr+24r^2+2n+12r}}{[q^6;q^6]_r\,[q^2;q^2]_n\,[q^2;q^2]_{2n+6r+2}\,[-q^3;q^3]_{2r+1}}$$

$$= \prod_{n\not\equiv 0,15,42 \pmod{57}} (1-q^n)^{-1} + q^4 \prod_{n\not\equiv 0,3,54 \pmod{57}} (1-q^n)^{-1}. \tag{8.12}$$

Next, in (8.6) setting $a=q^6$, $p=0$ and using (8.7), we have

$$[-q;q]_\infty \left\{1 + \sum_{\substack{r=0\\ (r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r-1}\, q^{2n^2+12nr+24r^2+6r}}{[q^6;q^6]_r\,[q^2;q^2]_n\,[q^2;q^2]_{2n+6r-1}\,[-q^3;q^3]_{2r}}\right\}$$

$$= \prod_{n\not\equiv 0,24,33 \pmod{57}} (1-q^n)^{-1}. \tag{8.13}$$

However, using (8.13) in (8.8), we get

$$[-q;q]_\infty \sum_{\substack{r=0\\ (r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r-1}(1-q^{2n+12r})\, q^{2n^2+12nr+24r^2-2n-6r}}{[q^6;q^6]_r\,[q^2;q^2]_n\,[q^2;q^2]_{2n+6r-1}\,[-q^3;q^3]_{2r}}$$

$$= \prod_{n\not\equiv 0,21,36 \pmod{57}} (1-q^n)^{-1}. \tag{8.14}$$

In (8.6) setting $a = q^6$, $p = 1$ and using (8.7), (8.8), we obtain

$$[-q;q]_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r} q^{2n^2+12nr+24r^2+2n+12r}}{[q^6;q^6]_r [q^2;q^2]_n [q^2;q^2]_{2n+6r+1} [-q^3;q^3]_{2r}}$$

$$= \prod_{n \not\equiv 0,18,39 (\mathrm{mod}\ 57)} (1-q^n)^{-1}. \tag{8.15}$$

In (8.6) setting $a = q^9$, $p = 0$ and using (8.9), we have

$$[-q;q]_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r}(1+q^{6r+3})\, q^{2n^2+12nr+24r^2+6n+24r}}{[q^6;q^6]_r [q^2;q^2]_n [q^2;q^2]_{2n+6r+2} [-q^3;q^3]_{2r+1}}$$

$$= \prod_{n \not\equiv 0,6,51 (\mathrm{mod}\ 57)} (1-q^n)^{-1}. \tag{8.16}$$

Lastly, in (8.6) setting $a = q^9$, $p = 1$ and using (8.10), we get

$$[-q;q]_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r}(1+q^{6r+3})\, q^{2n^2+12nr+24r^2+4n+18r}}{[q^6;q^6]_r [q^2;q^2]_n [q^2;q^2]_{2n+6r+2} [-q^3;q^3]_{2r+1}}$$

$$= \prod_{n \not\equiv 0,12,45 (\mathrm{mod}\ 57)} (1-q^n)^{-1}. \tag{8.17}$$

Identities of Rogers–Ramanujan Type Related to the modulus 51

In (8.1), letting x, $y \to \infty$ and $b \to 0$, we have

$$[a^2q^2;q^2]_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a^2;q^6]_{n+2r}(-)^r a^{2n+8r}\, q^{2n^2+12nr+27r^2-2np-6rp}}{[q^6;q^6]_r [q^2;q^2]_n [a^2;q^2]_{2n+6r} [-aq^3;q^3]_{2r}}$$

$$= \sum_{j=0}^{p} \frac{[q^{-2p};q^2]_j (-)^j a^{2j} q^{j(j+1)}}{[q^2;q^2]_j} \sum_{n=0}^{\infty} \frac{[a;q^3]_n (1-aq^{6n})(-)^n a^{8n}}{[q^3;q^3]_n (1-a)}$$

$$\cdot\, q^{(51/2)n^2-(3/2)n+12nj-6np}. \tag{8.18}$$

Transformation (8.18) for $a = 1$ and $p = 0, 1, 2$ yields

$$[-q;q]_\infty \left\{ 1 + \sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^6;q^6]_{n+2r-1}(-)^r q^{2n^2+12nr+27r^2}}{[q^6;q^6]_r [q^2;q^2]_n [q^2;q^2]_{2n+6r-1} [-q^3;q^3]_{2r}} \right\}$$

$$= \prod_{n \not\equiv 0,24,27 (\mathrm{mod}\ 51)} (1-q^n)^{-1}, \tag{8.19}$$

$$[-q;q]_\infty \left\{1+\sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r-1}(-)^r q^{2n^2+12nr+27r^2-2n-6r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r-1}[-q^3;q^3]_{2r}}\right\}$$

$$= \prod_{n\not\equiv 0,18,33 (\mathrm{mod}\ 51)}(1-q^n)^{-1} + \prod_{n\not\equiv 0,21,30 (\mathrm{mod}\ 51)}(1-q^n)^{-1} \qquad (8.20)$$

and

$$[-q;q]_\infty \left\{1+\sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r-1}(-)^r q^{2n^2+12nr+27r^2-4n-12r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r-1}[-q^3;q^3]_{2r}}\right\}$$

$$= \prod_{n\not\equiv 0,12,39 (\mathrm{mod}\ 51)}(1-q^n)^{-1} + (1+q^{-2})\prod_{n\not\equiv 0,24,27 (\mathrm{mod}\ 51)}(1-q^n)^{-1}$$

$$+ \prod_{n\not\equiv 0,15,36 (\mathrm{mod}\ 51)}(1-q^n)^{-1}, \qquad (8.21)$$

respectively.

On the other hand (8.18) for $a=q^3$ and $p=0$, 1 reduces to

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r}(-)^r q^{2n^2+12nr+27r^2+6n+24r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r+2}[-q^3;q^3]_{2r+1}}$$

$$= \prod_{n\not\equiv 0,3,48 (\mathrm{mod}\ 51)}(1-q^n)^{-1}, \qquad (8.22)$$

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r}(-)^r q^{2n^2+12nr+27r^2+4n+18r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r+2}[-q^3;q^3]_{2r+1}}$$

$$= \prod_{n\not\equiv 0,9,42 (\mathrm{mod}\ 51)}(1-q^n)^{-1} - q^3\prod_{n\not\equiv 0,3,48 (\mathrm{mod}\ 51)}(1-q^n)^{-1}. \qquad (8.23)$$

Now, using (8.22) in (8.23), we have

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}$$

$$\cdot\frac{[q^6;q^6]_{n+2r}(1+q^{2n+6r+3})(-)^r q^{2n^2+12nr+27r^2+4n+18r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r+2}[-q^3;q^3]_{2r+1}}$$

$$= \prod_{n\not\equiv 0,9,42 (\mathrm{mod}\ 51)}(1-q^n)^{-1}. \qquad (8.24)$$

Next, in (8.18) setting $a=q^6$, $p=0$ and using (8.19), we get

$$[-q;q]_\infty \left\{1+\sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r-1}(-)^r q^{2n^2+12nr+27r^2-6r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r-1}[-q^3;q^3]_{2r}}\right\}$$

$$=\prod_{n\not\equiv 0,21,30 \pmod{51}}(1-q^n)^{-1}. \tag{8.25}$$

However, using (8.25) in (8.20), we obtain

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r}(-)^r q^{2n^2+12nr+27r^2+6r+2n}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r+1}[-q^3;q^3]_{2r}}$$

$$=\prod_{n\not\equiv 0,18,33 \pmod{51}}(1-q^n)^{-1}. \tag{8.26}$$

Transformation (8.18) for $a=q^6$, $p=1$ in view of (8.19), (8.20) yields

$$[-q;q]_\infty \sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}$$

$$\cdot\frac{[q^6;q^6]_{n+2r-1}(1-q^{2n+12r})(-)^r q^{2n^2+12nr+27r^2-2n-12r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r-1}[-q^3;q^3]_{2r}}$$

$$=\prod_{n\not\equiv 0,15,36 \pmod{51}}(1-q^n)^{-1}. \tag{8.27}$$

Using (8.19) and (8.27), in (8.21) we obtain

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r}(-)^r q^{2n^2+12nr+27r^2-2}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r+1}[-q^3;q^3]_{2r}}$$

$$=\prod_{n\not\equiv 0,12,39 \pmod{51}}(1-q^n)^{-1}+q^{-2}\prod_{n\not\equiv 0,24,27 \pmod{51}}(1-q^n)^{-1}. \tag{8.28}$$

Lastly, in (8.18) setting $a=q^9$, $p=0$ and using (8.22), we have

$$[-q;q]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r}(-)^r(1+q^{6r+3})\,q^{2n^2+12nr+27r^2+6n+18r}}{[q^6;q^6]_r[q^2;q^2]_n[q^2;q^2]_{2n+6r+2}[-q^3;q^3]_{2r+1}}$$

$$=\prod_{n\not\equiv 0,6,45 \pmod{51}}(1-q^n)^{-1}. \tag{8.29}$$

Identities of Rogers–Ramanujan Type Related to the modulus 39

In (8.1) setting $y=-aq$ and letting $b, x \to \infty$, we get

$$\frac{[a^2q^2;q^2]_\infty}{[-aq;q^2]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \cdot\frac{[a^2;q^6]_{n+2r}[-aq;q^2]_{n+3r}a^{n+5r}q^{n^2+6nr+15r^2-2pn-6pr}}{[q^6;q^6]_r[q^2;q^2]_n[-aq^3;q^3]_{2r}[a^2;q^2]_{2n+6r}}$$
$$=\sum_{j=0}^{p}\frac{[q^{-2p};q^2]_j(-aq)^j}{[q^2;q^2]_j}\sum_{n=0}^{\infty}\frac{[a;q^3]_n(1-aq^{6n})[-aq;q^2]_{3n+j}}{[q^3;q^3]_n(1-a)[-aq;q^2]_{3n}} \cdot(-)^n a^{6n}q^{(39/2)n^2-(3/2)n-4n+6nj}. \tag{8.30}$$

Transformation (8.30) for $p=0$ and $a=1$, q^3 reduces to the following two identities

$$[-q^2;q^2]_\infty\left\{1+\sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r-1}[-q;q^2]_{n+3r}q^{n^2+6nr+15r^2}}{[q^6;q^6]_r[q^2;q^2]_n[-q^3;q^3]_{2r}[q^2;q^2]_{2n+6r-1}}\right\}$$
$$=\prod_{n\not\equiv 0,18,21 \pmod{39}}(1-q^n)^{-1}, \tag{8.31}$$

$$[-q;q^2]_\infty\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r}[-q^2;q^2]_{n+3r+1}q^{n^2+6nr+15r^2+3n+15r}}{[q^6;q^6]_r[q^2;q^2]_n[-q^3;q^3]_{2r+1}[q^2;q^2]_{2n+6r+2}}$$
$$=\prod_{n\not\equiv 0,3,36 \pmod{39}}(1-q^n)^{-1}. \tag{8.32}$$

On the other hand, (8.30) for $p=0$, $a=q^6$ in view of (8.31) yields

$$[-q^2;q^2]_\infty\left\{1+\sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\frac{[q^6;q^6]_{n+2r-1}[-q;q^2]_{n+3r}q^{n^2+6nr+15r^2+6r}}{[q^6;q^6]_r[q^2;q^2]_n[-q^3;q^3]_{2r}[q^2;q^2]_{2n+6r-1}}\right\}$$
$$=\prod_{n\not\equiv 0,15,24 \pmod{39}}(1-q^n)^{-1}. \tag{8.33}$$

Whereas in (8.30) setting $p=0$, $a=q^9$ and using (8.32), we have

$$[-q;q^2]_\infty\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \cdot\frac{[q^6;q^6]_{n+2r}[-q^2;q^2]_{n+3r+1}(1+q^{6r+3})q^{n^2+6nr+15r^2+3n+15r}}{[q^6;q^6]_r[q^2;q^2]_n[-q^3;q^3]_{2r+1}[q^2;q^2]_{2n+6r+2}}$$
$$=\prod_{n\not\equiv 0,6,33 \pmod{39}}(1-q^n)^{-1}. \tag{8.34}$$

Next, in (8.30) setting $p=1$, $a=1$ and using (8.33), we obtain

$$[-q^2;q^2]_\infty \sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \cdot \frac{[q^6;q^6]_{n+2r-1}[-q;q^2]_{n+3r}(1-q^{2n+12r})\,q^{n^2+6nr+15r^2-2n-6r}}{[q^6;q^6]_r[q^2;q^2]_n[-q^3;q^3]_{2r}[q^2;q^2]_{2n+6r-1}}$$

$$= \prod_{n\not\equiv 0,12,27 (\mathrm{mod}\ 39)} (1-q^n)^{-1} + q^{-1} \prod_{n\not\equiv 0,18,21 (\mathrm{mod}\ 39)} (1-q^n)^{-1}. \tag{8.35}$$

Lastly, in (8.30) setting $p=1$, $a=q^3$ and using (8.32), we obtain

$$[-q;q^2]_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \cdot \frac{[q^6;q^6]_{n+2r}[-q^2;q^2]_{n+3r+1}(1-q^{2n+6r+2})\,q^{n^2+6nr+15r^2+n+9r}}{[q^6;q^6]_r[q^2;q^2]_n[-q^3;q^3]_{2r+1}[q^2;q^2]_{2n+6r+2}}$$

$$= \prod_{n\not\equiv 0,9,30 (\mathrm{mod}\ 39)} (1-q^n)^{-1} - q^3 \prod_{n\not\equiv 0,3,36 (\mathrm{mod}\ 39)} (1-q^n)^{-1}. \tag{8.36}$$

Identities of Rogers–Ramanujan Type Related to the modulus 114

Transformation (8.2) for $b, x, y \to \infty$ becomes

$$[aq;q]_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a;q^3]_{n+2r}\,a^{n+4r}q^{n^2+6nr+12r^2-pn-3pr}}{[q^3;q^3]_r[[q;q]_n[aq^3;q^6]_r[a;q]_{2n+6r}}$$

$$= \sum_{j=0}^{p} \frac{[q^{-p};q]_j(-a)^j q^{(j/2)(j+1)}}{[q;q]_j} \cdot \sum_{n=0}^{\infty} \frac{[a;q^6]_n(1-aq^{12n})(-)^n a^{9n} q^{57n^2-3n+12nj-6np}}{[q^6;q^6]_n(1-a)}. \tag{8.37}$$

However, (8.37) for $a=1$ and $p=0$, 1 yields

$$1+\sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r-1}\,q^{n^2+6nr+12r^2}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_r[q;q]_{2n+6r-1}}$$

$$= \prod_{n\not\equiv 0,54,60 (\mathrm{mod}\ 114)} (1-q^n)^{-1}, \tag{8.38}$$

$$1+\sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r-1}\,q^{n^2+6nr+12r^2-n-3r}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_r[q;q]_{2n+6r-1}}$$

$$= \prod_{n\not\equiv 0,48,66 (\mathrm{mod}\ 114)} (1-q^n)^{-1} + \prod_{n\not\equiv 0,54,60 (\mathrm{mod}\ 114)} (1-q^n)^{-1}. \tag{8.39}$$

Using (8.38) in (8.39), we have

$$\sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r-1}(1-q^{n+3r})\,q^{n^2+6nr+12r^2-n-3r}}{[q^3;q^3]_r\,[q;q]_n\,[q^3;q^6]_r\,[q;q]_{2n+6r-1}}$$

$$= \prod_{n\not\equiv 0,48,66 (\mathrm{mod}\ 114)} (1-q^n)^{-1}. \tag{8.40}$$

Whereas, in (8.37) setting $a=1$, $p=2$ and using (8.39), we get

$$\sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r-1}(1-q^{n+3r})\,q^{n^2+6nr+12r^2-2n-6r}}{[q^3;q^3]_r\,[q;q]_n\,[q^3;q^6]_r\,[q;q]_{2n+6r-1}}$$

$$= \prod_{n\not\equiv 0,42,72 (\mathrm{mod}\ 114)} (1-q^n)^{-1} + q^{-1} \prod_{n\not\equiv 0,54,60 (\mathrm{mod}\ 114)} (1-q^n)^{-1}. \tag{8.41}$$

On the other hand, (8.37) for $a=q^6$ and $p=0$, 1 yields

$$\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r+1}\,q^{n^2+6nr+12r^2+6n+24r}}{[q^3;q^3]_r\,[q;q]_n\,[q^3;q^6]_{r+1}\,[q;q]_{2n+6r+5}}$$

$$= \prod_{n\not\equiv 0,6,108 (\mathrm{mod}\ 114)} (1-q^n)^{-1}, \tag{8.42}$$

$$\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r+1}\,q^{n^2+6nr+12r^2+5n+21r}}{[q^3;q^3]_r\,[q;q]_n\,[q^3;q^6]_{r+1}\,[q;q]_{2n+6r+5}}$$

$$= \prod_{n\not\equiv 0,12,102 (\mathrm{mod}\ 114)} (1-q^n)^{-1}. \tag{8.43}$$

However, in (8.37) setting $a=q^6$, $p=2$ and using (8.42), we get

$$\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r+1}\,q^{n^2+6nr+12r^2+4n+18r}}{[q^3;q^3]_r\,[q;q]_n\,[q^3;q^6]_{r+1}\,[q;q]_{2n+6r+4}}$$

$$= \prod_{n\not\equiv 0,18,96 (\mathrm{mod}\ 114)} (1-q^n)^{-1}. \tag{8.44}$$

Similarly, in (8.37) setting $a=q^6$, $p=3$ and using (8.43), we obtain

$$\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r+1}\,q^{n^2+6nr+12r^2+3n+15r}}{[q^3;q^3]_r\,[q;q]_n\,[q^3;q^6]_{r+1}\,[q;q]_{2n+6r+4}}$$

$$= \prod_{n\not\equiv 0,24,90 (\mathrm{mod}\ 114)} (1-q^n)^{-1} + q^4 \prod_{n\not\equiv 0,12,102 (\mathrm{mod}\ 114)} (1-q^n)^{-1}. \tag{8.45}$$

Identities of Rogers–Ramanujan Type Related to the modulus 102

In (8.2) letting $x, y \to \infty$, $b \to 0$, we have

$$[aq;q]_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a;q^3]_{n+2r} a^{n+4r} q^{n^2+6nr+(21/2)r^2-3r/2-pn-3pr}}{[q^3;q^3]_r [q;q]_n [a;q]_{2n+6r} [aq^3;q^6]_r}$$
$$= \sum_{j=0}^{p} \frac{[q^{-p};q]_j (-a)^j q^{(j/2)(j+1)}}{[q;q]_j}$$
$$\cdot \sum_{n=0}^{\infty} \frac{[a;q^6]_n (1-aq^{12n})(-)^n a^{8n} q^{51n^2-3n+12nj-6np}}{[q^6;q^6]_n (1-a)}. \qquad (8.46)$$

Transformation (8.46) for $a=1$ and $p=0$, 1 yields

$$1 + \sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r-1} q^{n^2+6nr+(21/2)r^2-(3/2)r}}{[q^3;q^3]_r [q;q]_n [q^3;q^6]_r [q;q]_{2n+6r-1}}$$
$$= \prod_{n \not\equiv 0,48,54 (\mathrm{mod}\ 102)} (1-q^n)^{-1}, \qquad (8.47)$$

$$1 + \sum_{\substack{r=0 \\ (n,r)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r-1} q^{n^2+6nr+(21/2)r^2-n-(9/2)r}}{[q^3;q^3]_r [q;q]_n [q^3;q^6]_r [q;q]_{2n+6r-1}}$$
$$= \prod_{n \not\equiv 0,42,60 (\mathrm{mod}\ 102)} (1-q^n)^{-1} + \prod_{n \not\equiv 0,48,54 (\mathrm{mod}\ 102)} (1-q^n)^{-1}. \qquad (8.48)$$

However, using (8.47) in (8.48), we get

$$\sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r-1}(1-q^{n+3r})\, q^{n^2+6nr+(21/2)r^2-n-(9/2)r}}{[q^3;q^3]_r [q;q]_n [q^3;q^6]_r [q;q]_{2n+6r-1}}$$
$$= \prod_{n \not\equiv 0,42,60 (\mathrm{mod}\ 102)} (1-q^n)^{-1}. \qquad (8.49)$$

In (8.46) setting $a=1$, $p=2$ and in view of (8.48), we have

$$\sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \frac{[q^3;q^3]_{n+2r-1}(1-q^{n+3r})\, q^{n^2+6nr+(21/2)r^2-2n-(15/2)r}}{[q^3;q^3]_r [q;q]_n [q^3;q^6]_r [q;q]_{2n+6r-1}}$$
$$= \prod_{n \not\equiv 0,36,66 (\mathrm{mod}\ 102)} (1-q^n)^{-1} + q^{-1} \prod_{n \not\equiv 0,48,54 (\mathrm{mod}\ 102)} (1-q^n)^{-1}. \qquad (8.50)$$

Next, (8.46) for $a=q^6$ and $p=0$, 1, gives

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^3;q^3]_{n+2r+1}q^{n^2+6nr+(21/2)r^2+6n+(45/2)r}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_{r+1}[q;q]_{2n+6r+5}}$$
$$=\prod_{n\not\equiv 0,6,96(\bmod 102)}(1-q^n)^{-1}. \tag{8.51}$$

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^3;q^3]_{n+2r+1}q^{n^2+6nr+(21/2)r^2+5n+(39/2)r}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_{r+1}[q;q]_{2n+6r+5}}$$
$$=\prod_{n\not\equiv 0,12,90(\bmod 102)}(1-q^n)^{-1}. \tag{8.52}$$

On the other hand, in (8.46) setting $a=q^6$, $p=2$ and using (8.51), we get

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^3;q^3]_{n+2r+1}q^{n^2+6nr+(21/2)r^2+4n+(33/2)r}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_{r+1}[q;q]_{2n+6r+4}}$$
$$=\prod_{n\not\equiv 0,18,84(\bmod 102)}(1-q^n)^{-1}. \tag{8.53}$$

Similarly, in (8.46) setting $a=q^6$, $p=3$ and using (8.52), we have

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^3;q^3]_{n+2r+1}q^{n^2+6nr+(21/2)r^2+3n+(27/2)r}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_{r+1}[q;q]_{2n+6r+4}}$$
$$=\prod_{n\not\equiv 0,24,78(\bmod 102)}(1-q^n)^{-1}+q^4\prod_{n\not\equiv 0,12,90(\bmod 102)}(1-q^n)^{-1}. \tag{8.54}$$

Identities of Rogers–Ramanujan Type Related to the modulus 78

In (8.2) setting $p=0$, $y=-\sqrt{aq}$ and letting b, $x\to\infty$, we get

$$\frac{[aq;q]_\infty}{[-\sqrt{aq};q]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[-\sqrt{aq};q]_{n+3r}[a;q^3]_{n+2r}a^{n/2+(5r/2)}q^{(n^2/2)+3nr+(15/2)r^2}}{[q^3;q^3]_r[q;q]_n[a;q]_{2n+6r}[aq^3;q^6]_r}$$
$$=\sum_{n=0}^{\infty}\frac{[a;q^6]_n(1-aq^{12n})(-)^n a^{6n}q^{39n^2-3n}}{[q^6;q^6]_n(1-a)}. \tag{8.55}$$

However, (8.55) for $a=1$, q^6 and q^{12} yields the following three identities

$$\frac{1}{[-\sqrt{q};q]_\infty}\left\{1+\sum_{\substack{r=0\\(r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\right.$$
$$\left.\cdot\frac{[q^3;q^3]_{n+2r-1}[-\sqrt{q};q]_{n+3r}q^{n^2/2+3nr+(15/2)r^2}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_r[q;q]_{2n+6r-1}}\right\}$$
$$=\prod_{n\not\equiv 0,36,42(\bmod 78)}(1-q^n)^{-1}, \tag{8.56}$$

$$\frac{1}{[-\sqrt{q};q]}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\cdot\frac{[q^3;q^3]_{n+2r+1}[-\sqrt{q};q]_{n+3r+3}q^{(1/2)n^2+3nr+(15/2)r^2+3n+15r}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_{r+1}[q;q]_{2n+6r+5}}$$

$$=\prod_{n\not\equiv 0,6,72(\text{mod } 78)}(1-q^n)^{-1} \tag{8.57}$$

and

$$\frac{1}{[-\sqrt{q};q]}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\cdot\frac{[q^3;q^3]_{n+2r+3}[-\sqrt{q};q]_{n+3r+6}q^{(1/2)n^2+3nr+(15/2)r^2+6n+30r+30}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_{r+2}[q;q]_{2n+6r+11}}$$

$$=\prod_{n\not\equiv 0,36,42(\text{mod } 78)}(1-q^n)^{-1}-\prod_{n\not\equiv 0,30,48(\text{mod } 78)}(1-q^n)^{-1}. \tag{8.58}$$

Identities of Rogers–Ramanujan Type Related to the modulus 66

In (8.2) setting $p=0$, $y=-\sqrt{aq}$ and letting $x\to\infty$, $b\to 0$, we obtain

$$\frac{[aq;q]_\infty}{[-\sqrt{aq};q]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\cdot\frac{[a;q^3]_{n+2r}[-\sqrt{aq};q]_{n+3r}a^{n/2+(5r/2)}q^{(1/2)n^2+3nr+9r^2-(3/2)r}}{[q^3;q^3]_r[q;q]_n[aq^3;q^6]_r[a;q]_{2n+6r}}$$

$$=\sum_{n=0}^{\infty}\frac{[a;q^6]_n(1-aq^{6n})a^{5n}(-)^n q^{33n^2-3n}}{[q^6;q^6]_n(1-a)}. \tag{8.59}$$

However, (8.59) for $a=1$, q^6 and q^{12} give the following three identities:

$$\frac{1}{[-\sqrt{q};q]_\infty}\left\{1+\sum_{\substack{r=0 \\ (r,n)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty}\cdot\frac{[q^3;q^3]_{n+2r-1}[-\sqrt{q};q]_{n+3r}q^{(1/2)n^2+3nr+9r^2-(3/2)r}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_r[q;q]_{2n+6r-1}}\right\}$$

$$=\prod_{n\not\equiv 0,30,36(\text{mod } 66)}(1-q^n)^{-1}, \tag{8.60}$$

$$\frac{1}{[-\sqrt{q};q]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\cdot\frac{[q^3;q^3]_{n+2r+1}[-\sqrt{q};q]_{n+3r+3}q^{(1/2)n^2+3nr+9r^2+3n+(27/2)r}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_{r+1}[q;q]_{2n+6r+5}}$$

$$=\prod_{n\not\equiv 0,6,60 \pmod{66}}(1-q^n)^{-1} \tag{8.61}$$

and

$$\frac{1}{[-\sqrt{q};q]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\cdot\frac{[q^3;q^3]_{n+2r+2}[-\sqrt{q};q]_{n+3r+6}q^{n^2/2+3nr+9r^2+6n+(57/2)r+24}}{[q^3;q^3]_r[q;q]_n[q^3;q^6]_{r+2}[q;q]_{2n+6r+11}}$$

$$=\prod_{n\not\equiv 0,30,36 \pmod{66}}(1-q^n)^{-1}-\prod_{n\not\equiv 0,24,42 \pmod{66}}(1-q^n)^{-1}. \tag{8.62}$$

REFERENCES

1. R. P. AGARWAL, On integral analogues of certain transformations of well-poised basic hypergeometric series, *Quart. J. Math. Oxford Ser. (2)* **4** (1953), 161–167.
2. R. P. AGARWAL AND A. VERMA, Generalized basic hypergoemetric series with unconnected bases, *Proc. Cambridge Philos. Soc.* **63** (1967), 727–734.
3. G. E. ANDREWS, On Rogers–Ramanujan type identities related to the modulus 11, *Proc. London Math. Soc. (3)* **30** (1975), 330–346.
4. G. E. ANDREWS, Plane partitions. III. The weak Macdonald conjecture, *Invent. Math.* **53** (1979), 193–225.
5. W. N. BAILEY, "Generalized Hypergeometric Series," Cambridge Tract, 1935.
6. W. N. BAILEY, A transformation of nearly-poised basic hypergeometric series, *J. London Math. Soc.* **22** (1947), 237–240.
7. W. N. BAILEY, Some identities in combinatory analysis, *Proc. London Math. Soc. (2)* **49** (1947), 421–435.
8. W. N. BAILEY, Identities of the Rogers–Ramanujan type, *Proc. London Math. Soc. (2)* **50** (1949), 1–10.
9. V. K. JAIN, Certain transformations of basic hypergeometric series and their applications, *Pacific J. Math.*, in press.
10. I. G. MACDONALD, Affine root system and Dedekind's η-function, *Invent. Math.* **15** (1972), 91–143.
11. D. B. SEARS, On the transformation theory of hypergeometric functions, *Proc. London Math. Soc. (2)* **52** (1950), 14–35.
12. D. B. SEARS, Transformations of basic hypergeometric functions of special type, *Proc. London Math. Soc. (2)* **52** (1951), 467–483.
13. D. B. SEARS, On the transformation theory of hypergeometric functions and cognate trigonometrical series, *Proc. London Math. Soc. (2)* **53** (1951), 138–157.

14. D. B. SEARS, On the transformation theory of basic hypergeometric functions, *Proc. London Math. Soc. (2)* **53** (1951), 150–180.
15. D. B. SEARS, Transformations of basic hypergeometric functions of any order, *Proc. London Math. Soc. (2)* **53** (1951), 181–191.
16. L. J. SLATER, A new proof of Rogers transformations of infinite series, *Proc. London Math. Soc. (2)* **53** (1951), 460–475.
17. L. J. SLATER, Further identities of the Rogers–Ramanujan type, *Proc. London Math. Soc. (2)* **54** (1952), 147–167.
18. L. J. SLATER, Integrals representing general hypergeometric transformations, *Quart. J. Math. Oxford Ser. (2)* **3** (1952), 206–216.
19. L. J. SLATER, "Generalized Hypergeometric Functions," Cambridge Univ. Press, London/New York, 1966.
20. A. VERMA, Certain summation formulae for basic hypergeometric series, *Canad. Math. Bull.* **20** (1977), 369-375.
21. A. VERMA AND V. K. JAIN, Transformations between basic hypergeometric series on different bases and identities of Rogers–Ramanujan type, *J. Math. Anal. Appl.* **76** (1980) 230–269.
22. A. VERMA AND V. K. JAIN, q-Analogue of a transformation of Whipple, Communicated for publication.
23. G. N. WATSON, The continuation of functions defined by generalized hypergeometric series, *Trans. Cambridge Philos. Soc.* **11** (1910), 281–299.
24. G. N. WATSON, A new proof of the Rogers–Ramanujan identities, *J. London Math. Soc.* **4** (1929), 4–9.
25. F. J. W. WHIPPLE, Some transformations of generalized hypergeometric series, *Proc. London Math. Soc. (2)* **26** (1927), 257–272.